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Evaluation of single-prioritization voting systems in controlled collective Parrondo's games

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Abstract We examine an ensemble of individuals playing three individually losing games. With three games to choose from, what happens when players are allowed to vote for the game that will be played by the entire ensemble? The collective choice cannot conform to the preference of all the players simultaneously. As such, a subset of players will have to implement strategies to force the outcome of the vote to favour their preferences. How does this affect the outcome of playing these three games? We show that for plurality voting, the introduction of the spoiler option significantly changes the outcome of Parrondo's games. Rank choice and approval voting are not susceptible to the spoiler effect. However, the complexity associated with rank choice and approval voting implies that other systemic flaws, such as the centre squeeze and the Volunteer's dilemma, emerge.

Keywords Controlled collective decision · Decision making · Single-prioritization voting · Parrondo's games

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1 Introduction

Voting is a key function in society to collectively decide, from a set of options, an action to be taken. Individuals in society have their own preferences from the set of options, which can alter outcomes if the decision is switched. When arriving at a decision, each individual can cast a ballot by indicating or listing one's preference. The outcome depends on the voting system used, but often seeks to aggregate towards the centre of consensus. There are several ways in which individuals or groups may identify the centre of consensus, giving rise to a gap between voting results and group opinion, which remains a rich field of study [1]. In the case of arriving at a single prioritization, common voting systems include: majority/plurality voting, rank choice voting, and approval voting. While it is in the interest of the individual to have one's preference chosen, it is practically impossible to satisfy all individuals in a society where there are a variety of preferences, especially in the case of ranked choice, as proven by Arrow's impossibility theorem [2]. While collective decision making is a major topic in economics and choice theory, there has been increasing interest to apply them in computer science, machine learning, game theory, and control theory [3–5].

Parrondo's paradox, an abstraction of the flashing Brownian ratchets [6–8], refers to the phenomenon where a winning outcome can be achieved by alternating between two losing strategies. A recent review discusses the applications of Parrondo's paradox in social settings [9] in which multiple losing options are combined in a certain manner to achieve a winning outcome. Parrondo's paradox has many applications, such as biology and ecology [10–12], where species undergo switching of survival strategies to maintain population growth; science and engineering [13–15], where stability is observed from unstable equilibrium through random, periodic, and chaotic switching, leading to useful development in encryption [16]; and emerging technologies like quantum information [17–21]. More recently, Parrondo's paradox has even been shown to be beneficial in epidemic control strategies [22]. The wide emergence of the paradox in a large number of fields reveals the far-reaching impact of the Parrondo's paradox. Research involving the cooperative variant of Parrondo's paradox [23] has been used to investigate sociodynamical systems. In particular, the paradox has also been observed in certain social trends [24,25], including efficient collective voting [26–28], expediting information spread [29], the matching problem [30], effective resource redistribution and management [23,31], and its potential to alleviate traffic congestion [32].

Consider two disadvantageous options, modelled as two losing games. The first, game A, a player gains or loses 1 unit of capital depending on the outcome of the game. If the player wins, a unit of capital is gained, otherwise, a unit of capital is lost. The probability of a win for the player, is $p_A = \frac{1}{2} - \epsilon$, with $0 \leq \epsilon \ll 1$. Game A is a winning, fair, or losing game depending on the value of ϵ. In the case where game A is modelled to be a losing game, $\epsilon > 0$.

The second game, game B, is also a losing game, depending on the existing capital of the player. If the player's capital is a multiple of three, then the probability of winning game B is $p_{B1} = \frac{1}{10} - \epsilon$, otherwise the probability of winning is $p_{B2} = \frac{3}{4} - \epsilon$. While not immediately apparent that game B is a losing game, it can be shown that game B is also a losing game for $\epsilon > 0$ [9]. The rules of these two games are illustrated in Fig. 1.

Parrondo's paradox tells us that by stochastically switching between games A and B or by following some periodic sequences, one can achieve a winning outcome for $\epsilon > 0$ being sufficiently small [33–35]. That is to say, on average, the player's capital grows with the number of turns, indicating that it is possible to achieve a winning outcome from two losing games. The switching in either case does not involve any decision input by the player to maximize one's capital. If, however, the player can control which game to play, the problem becomes trivial. The best strategy is to choose to play game A when one's capital is a multiple of three, and play game B, otherwise. What happens if this counter-intuitive phenomenon is played among an ensemble of individuals as controlled collective games? It is no longer obvious how this will affect the outcome of the collective games as the choice of playing the games collectively will benefit some individuals but not others. In several works by Dinís and Parrondo et al. [26–28], they showed that blind strategies are winning, whereas strategies which choose the game with the highest average return and collective decision played through majority voting, are both losing. Dictatorial decision making is beneficial, and the outcome of decision-making oligarchies is inefficient compared to collective decision making by the entire ensemble.

In this paper, we investigate the possibility of controlled collective games in the setting of Parrondo's games by including decision-making protocols made through democratic choice. The voting schemes discussed in this paper include plurality, rank choice, and approval voting. We show that unlike plurality voting, rank choice and approval voting are beneficial in achieving winning outcomes in the 3-option setup. The paper is organized as follows: in Sect. 2, we introduce the third losing game and explain the rank choice of each game. In Sect. 3, we consider the various voting schemes and show how the outcome can be manipulated by exploiting the flaws of plurality and rank choice voting. In the same section, we further investigate optimization of approval voting. Lastly, we present our main conclusions in Sect. 4.

2 3-option controlled collective Parrondo's game

The third Parrondo's games for consideration is the history-dependent Parrondo's game. The history-dependent game, denoted as game C, is a Parrondo's game with winning outcomes dependent on the past performance of each individual. We denote the history of each player at each time step t as $(s_{t-1},\ s_{t-2})$, where $s_t \in \{\text{Win, Lose}\}$. For game C, the probability of winning game C is $p_{C1} = \frac{9}{10} - \epsilon$ if the player's history is (Lose, Lose), $p_{C2} = \frac{1}{4} - \epsilon$ if the player's history is

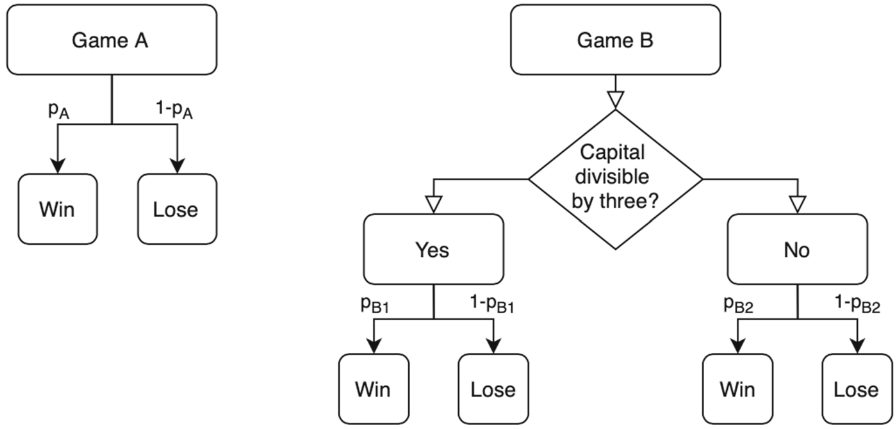


Fig. 1 Rules for games A and B. Game B is capital dependent

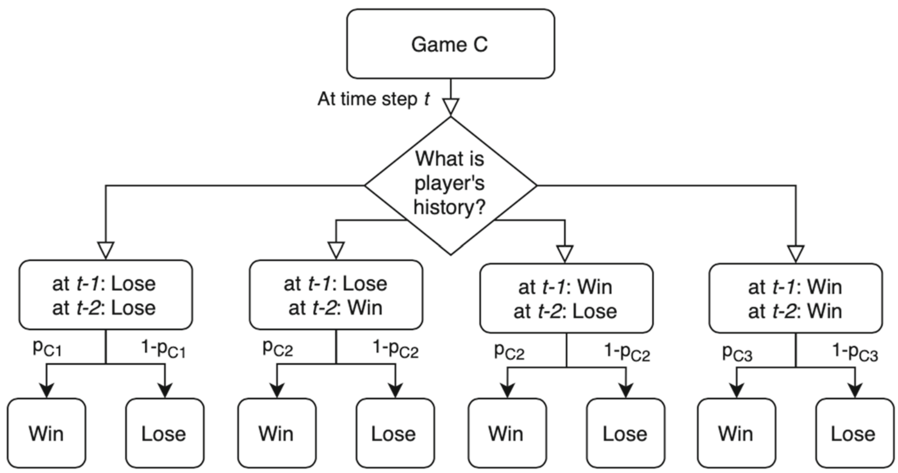


Fig. 2 Rules for game C. Game C is a history dependent

either (Lose, Win) or (Win, Lose), and $p_{C3} = \frac{7}{10} - \epsilon$ if the player's history is (Win, Win). The rules of game C are illustrated in Fig. 2. Again, while not immediately apparent, game C is also losing [9].

Now, consider the individual who is playing a controlled game, with three possible games to rank according to preference. The individual will have different preference rankings depending on the state that the individual is in. There are nine possible states, four historic states, i.e. (Lose, Lose), (Lose, Win), (Win, Lose), and (Win, Win), for capital divisible by three, and eight historic states for capital not divisible by three. The individual will rank the game according to the probability of winning with full information of one's own state. We denote "Game A $\succ$ Game B" if game A is preferred over game B. The list of preferences are:

- Capital divisible by 3 and history is (Lose, Lose) or (Win, Win): Game C $\succ$ Game A $\succ$ Game B
- Capital divisible by 3 and history is (Lose, Win) or (Win, Lose): Game A $\succ$ Game C $\succ$ Game B
- Capital not divisible by 3 and history is (Lose, Lose): Game C $\succ$ Game B $\succ$ Game A
- Capital not divisible by 3 and history is (Win, Win): Game B $\succ$ Game C $\succ$ Game A
- Capital not divisible by 3 and history is (Lose, Win) or (Win, Lose): Game B $\succ$ Game A $\succ$ Game C

The Markov chain with transition probabilities of the most preferred game at each state is illustrated in Fig. 3. As a shorthand, for the remainder of the paper, we will denote states with (C, H), where $C \in \{0, 1, 2\}$ denoting the capital modulo 3, and $H \in \{(L,L), (L,W), (W,L), (W,W)\}$ denoting the history of outcomes.

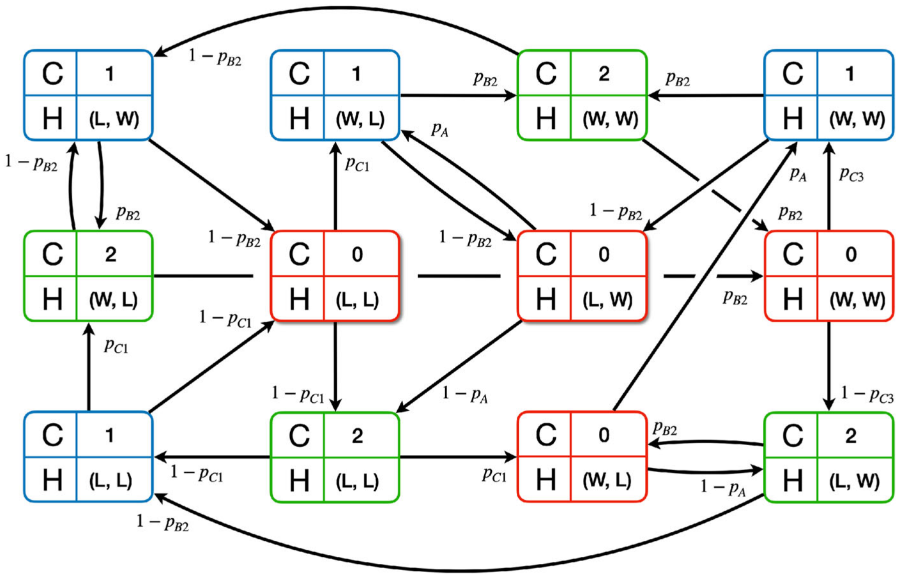


Fig. 3 Markov chain of states, where **C** is the capital modulo 3, and **H** is the history in the form $(s_{t-1},\ s_{t-2})$, where $s_t \in$ {L,W} at each time step t. The probabilities show the evolution if players in each state transit according to their most preferred game, and are located close to the arrow head. Notice that there are two ways to leave each state, either by winning or losing the preferred game; similarly, there are two ways of arriving at the state

Suppose there is a unique assignment of the alphabets $i = 1, \ldots, 12$ to the set of states $S = \{(C, H)\,|\,C \in \{0, 1, 2\},\ \ H \in \{(\text{L,L}), (\text{L,W}), (\text{W,L}), (\text{W,W})\}\}$. Analytically, the Markov chain in Fig. 3 can be written as an equation of detailed balance:

$$\boldsymbol{\pi}(t+1) = \Pi^{(k)} \boldsymbol{\pi}(t), \tag{1}$$

where $\boldsymbol{\pi}(t)$ is a column vector with 12 elements corresponding to the fraction of players in each of the states in S, and $\Pi^{(k)}$ is a 12×12 dynamic transition matrix, dependent on the game chosen, $k \in \{A, B, C\}$. The entries of $\Pi^{(k)}$ can be derived from Fig. 3 as follows:

$$\Pi^{(k)}_{ij} = \begin{cases} q_{j,i} & \text{if } \exists\ j \rightarrow i \\ 0 & \text{otherwise} \end{cases}, \tag{2}$$

where $q_{j,i}$ represents the transition probability from state j to state i, and $j \rightarrow i$ denotes an edge from j to i, satisfying

$$\sum_i \Pi^{(k)}_{ij} = 1,\ \forall j. \tag{3}$$

Equation (3) ensures that the total probability of transition from one state to all possible daughter states is 1. Since $\Pi^{(k)}$ is a sparse matrix, efficient computations can be performed numerically to verify results. For example, if the ensemble collectively plays game A, then $\Pi^{(A)}$ takes the form in Eq. (4). The q values for each column is assigned as follows: the state transiting $C \leftarrow C+1 \mod 3$ is $\frac{1}{2} - \epsilon$, and $C \leftarrow C-1 \mod 3$ is $\frac{1}{2} + \epsilon$.

$$\Pi = \begin{array}{c} \begin{array}{cccccccccccc} (0,(L,L)) & (0,(L,W)) & (0,(W,L)) & (0,(W,W)) & (1,(L,L)) & (1,(L,W)) & (1,(W,L)) & (1,(W,W)) & (2,(L,L)) & (2,(L,W)) & (2,(W,L)) & (2,(W,W)) \end{array} \\ \left(\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & q_{5,1} & q_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_{7,2} & q_{8,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{9,3} & q_{10,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{11,3} & q_{12,3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{9,5} & q_{10,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{11,6} & q_{12,6} \\ q_{1,7} & q_{2,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{3,8} & q_{4,8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_{1,9} & q_{2,9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{3,10} & q_{4,10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{5,11} & q_{6,11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_{7,12} & q_{8,12} & 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} (0,(L,L)) \\ (0,(L,W)) \\ (0,(W,L)) \\ (0,(W,W)) \\ (1,(L,L)) \\ (1,(L,W)) \\ (1,(W,L)) \\ (1,(W,W)) \\ (2,(L,L)) \\ (2,(L,W)) \\ (2,(W,L)) \\ (2,(W,W)) \end{array} \end{array} \tag{4}$$

3 Voting schemes

In a controlled collective game of N individuals, with randomly assigned initial capital and history, the preferred first choice of each individual differs according to the list above. In fact, in a randomly assigned initial state, where initial capital and history are assigned independently of each other, 1/6 of the individuals prefers to play game A, 1/2 prefers game B, and 1/3 prefers game C. We now examine how various single-prioritization voting systems affect the outcome of playing Parrondo's games.

3.1 Plurality voting

In the case of the plurality voting strategy, every player votes for the game that gives one the highest probability of winning. The game played by the entire ensemble of players is the one that receives the most votes. In the case of plurality voting, it does not necessarily require the majority of players to agree with the preference. In fact, the flaw with plurality voting is that it may not satisfy the preference of the majority—leading to potential capital loss. However, we know that if the ensemble comprises only a single agent, the controlled game can always be maximized. In the first simulation, we examine the outcome of choosing between three losing games for an ensemble of N individuals through plurality voting. Players will always vote for their top option. The results are shown in Fig. 4.

Clearly, the simulation results agree with the 2-option controlled collective results presented by Dinís and Parrondo [27], that under the 3-option plurality voting regime, an increasing number of players does lead to a losing outcome. Parrondo effect weakens with increasing number of individuals in the ensemble.

However, plurality voting is very susceptible to the spoiler effect. The spoiler effect could come in the form of a new option, which takes away votes from what would be an apparent majority choice. In the case of a 3-option controlled collective game, game A is a "conservative" game, where the probability of winning is close to the probability of losing, i.e. 0.495 and 0.505, respectively. Game A is a spoiler option. A player, in general, and in the case of our setup with state (0, (L,W)) or (0, (W,L)), does not risking much by choosing to play game A, as compared to the two other games. Suppose such a player switches strategy and chooses game C for its top choice, instead of game A, this changes the controlled collective game in two ways: (1) this now becomes a 2-option controlled collective game, as no player will vote for game A. The votes are split between games B and C. (2) Players in the state (0, (L,W)) or (0, (W,L)), reduce their chance of winning by about half, i.e. from 0.495 to 0.245. The results of such a simulation for $N = 5000$ are presented in Fig. 5.

Paradoxically, the combination of eliminating game A as the "spoiler" option and taking bigger risks by a subset of the ensemble results in a winning outcome for a 2-option controlled collective game. We show that it is possible to turn a losing outcome of a 3-option controlled collective game into a winning outcome by removing the "spoiler" option—game A.

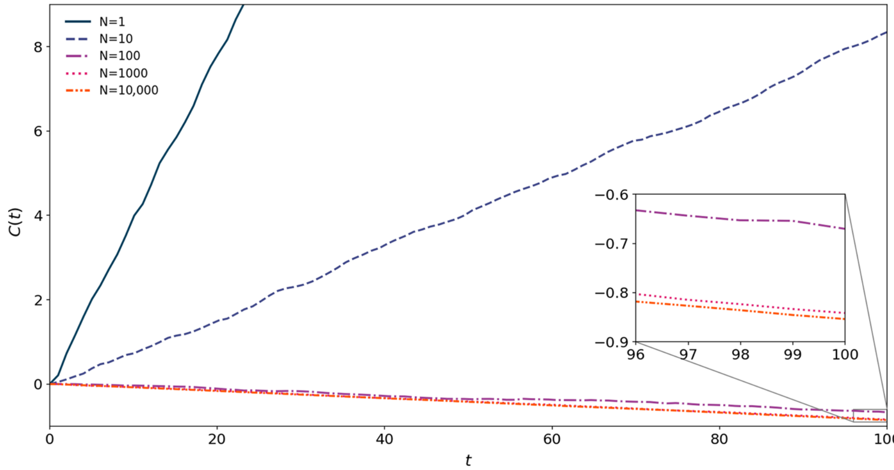


Fig. 4 A comparison of average capital per player after $t = 100$ time steps for an ensemble with $N = \{1, 10, 100, 1000, 10{,}000\}$ players playing 3-choice controlled collective game using plurality voting

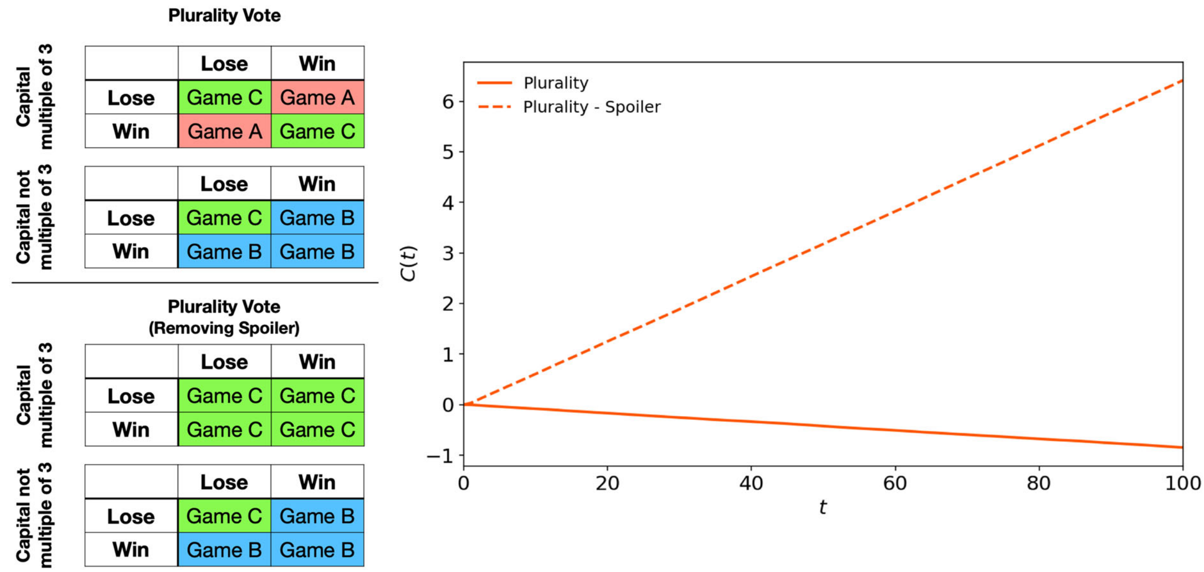


Fig. 5 A comparison of average capital per player after $t = 100$ time steps for an ensemble with $N = 5000$ players. The solid line is the outcome from the 3-option controlled collective game using plurality voting. The dash line is the outcome when players with game A as the top choice choose instead to list game C as the top choice; this is a 2-option controlled collective game, with players using plurality (or majority) voting between games B and C

3.2 Rank choice voting

A second form of voting to achieve single prioritization is through rank choice voting. As the name suggests, the ensemble of voters is mandated to rank all the options available in descending order. That is, in a ballot with m options, each option is assigned a unique rank by voter i, $R_i = 1, \ldots, m$, with 1 being the top choice. The total number of top choice given to each option is counted. At this point, if an option receives more than half the total number of ballots cast, that option is chosen as the collective decision, and all players in the ensemble play that game. Otherwise, if no option receives half the total number of ballots cast, the option that receives the least number of first choice votes is removed. The ballot count proceeds as if the option with the least top choice votes was not in the contest. This process is repeated until an option exceeds majority votes. This process has at most $m - 1$ iterations. The algorithm for rank choice voting is described in Fig. 6. We examine how this form of single-prioritization voting differs from plurality voting.

Similar to plurality voting, we examine the outcome of choosing between three losing games for an ensemble of N individuals through rank choice voting. The results are shown in Fig. 7. Like the results from the simulations of plurality voting, we observe that by increasing the number of voters, from $N = 1$ to $N = 10{,}000$, it frustrates the capital per player output, and the capital gained saturates. However, unlike its plurality counterpart, rank choice voting does not lead to average capital loss.

However, the disadvantage of rank choice voting is the effect of the centre squeeze [36]. In particular, "centrist" options, like game A, are usually squeezed out of the rank choice voting algorithm in the first round of voting. This may lead to strategic voting where voters "squeeze" game A out of the ballot, instead settling for game C as their first choice. This effectively results in a rank choice vote becoming a majority vote between games B and C. In separate simulations, we conduct an experiment comparing 3-option and 2-option (comprising of only games B and C) controlled collective game for $N = 5000$ players. The average capitals per player, at $t = 100$ are $C_{3\text{-option}} \approx 6.3969$ and $C_{2\text{-option}} \approx 6.3755$, respectively. It is clear that there is no significant difference between outcome of both simulations. In this simulation, the presence (or absence) of game A as an option does not significantly alter the outcome of the voting process. This is the centre

Fig. 6 Flow chart of rank choice voting

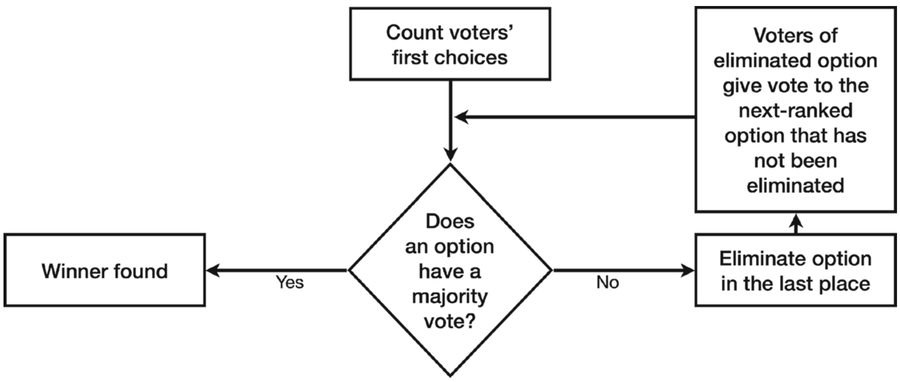
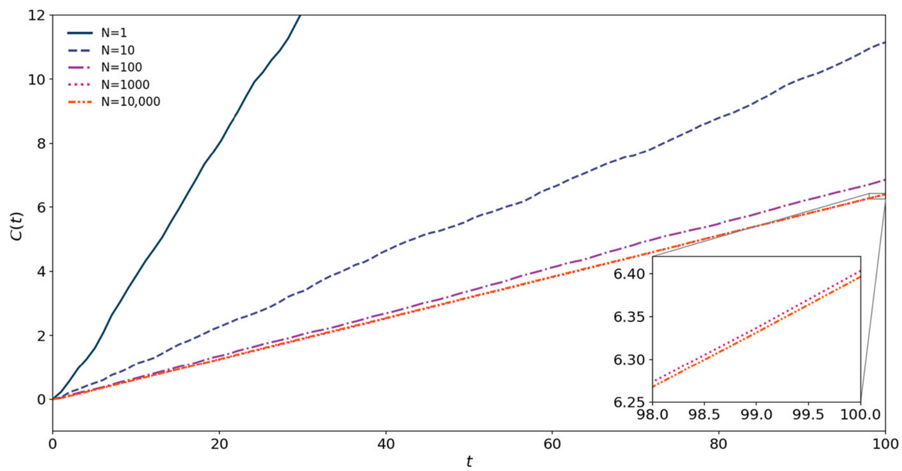


Fig. 7 A comparison of average capital per player after $t = 100$ time steps for an ensemble with $N = \{1, 10, 100, 1000, 10{,}000\}$ players playing 3-option controlled collective game using rank choice voting



squeeze effect of rank choice voting, where the "centrist" option, game A, is squeezed out of contention. In the 3-option version of rank choice voting, game A is always eliminated from the voting at the first round as it is preferred by the least number of voters, ultimately reducing this to a 2-option vote. Game A plays the role of a "centrist" option as in most cases, games B and C often pose the least risk to a majority of the voters. Game A is a safe option as there is a small difference in the win–lose probability. The "irrelevance" of game A is exacerbated by the fact that the risk involved in choosing game A is small, thus "squeezing" it out of the options does not significantly alter final outcomes.

3.3 Approval voting

Lastly, the final form of voting for achieving single prioritization that will be considered in this paper is approval voting. In approval voting, each voter indicates all the options that are favourable on the ballot. The voting can be concluded in a single round, where each vote of approval counts to the total number of votes for the option. The option with the most number of approvals is the collective decision made by the ensemble of voters. Approval voting allows for flexibility as it does not require voters to rank all the options, while at the same time, "centrist" options are not squeezed out of the vote. In this section, we present a framework for modelling approval voting according to the list of preferences introduced in Sect. 2. In our simulation, a voter is

(1) always approving of their top choice,
(2) approving of their second choice with a probability of r_2, and
(3) also approving of their third choice, given that they approve of their second choice with a probability of $r_{3|2}$.

The results of the simulation, using this framework, are shown in Fig. 8.

It is clear from Fig. 8 that approval voting is beneficial in 3-option controlled collective Parrondo's games

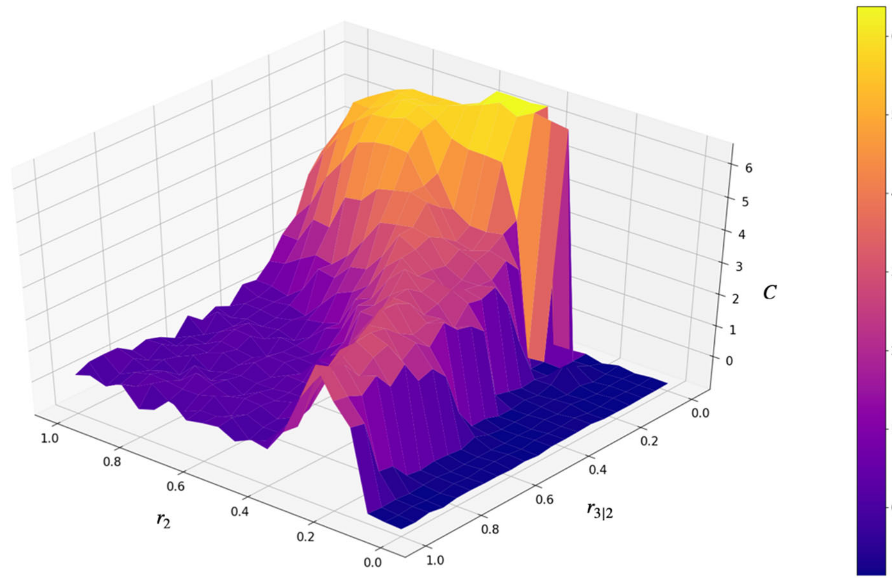


Fig. 8 A comparison of average capital per player at $t = 100$ time steps for an ensemble with $N = 5000$ players playing 3-option controlled collective game using approval voting. All voters approve of their first choice. The probability of approving of their second choice is r_2, and the probability of a voter approving their third choice given they approve their second choice is $r_{3|2}$

if a plurality of voters choose to exercise approval for their second choice. In the case where voters are selfish, by approving of their first choice only, the collective gain is significantly lower. It is indicative that the approval of all options by all voters is not as beneficial. In fact, the simulation reveals that the best outcome is only when 50% of players approve of their second choice, while at the same time, not approving of their third choice. This gives rise to a phenomenon called the *Volunteer's dilemma*, where total cooperation might lead to lesser gains, and not cooperating is mutually destructive [37]. Observing from Fig. 8, the peak of the graph ($r_2 = 0.5,\ r_{3|2} = 0$) gives us the best average capital outcome. Any deviation from that, either by total cooperation (or total fairness), analogous to allowing everyone to approve their second choice ($r_2 = 1$), or allowing everyone who voted for their second choice to also vote for their third choice ($r_2 = 1,\ r_{3|2} = 1$), we shift away from the best outcome. On the other extreme, by not cooperating, every voter decides on the best possible outcome for themselves by not giving an approval vote to other options ($r_2 = 0,\ r_{3|2} = 0$), it leads to the worst possible outcome.

4 Conclusion and future work

Our models have provided a framework for controlled collective games of more than two options. This is an extension of Parrondo's paradox, where typically only two losing games are considered. Here, by introducing three losing games, it shows the usefulness of various voting systems as a framework to vote for single prioritization. Voters in our framework are "controlled". They behave in a deterministic manner and vote for the option that maximizes individual outcome. However, multiple individuals could decide to vote to maximize group outcome instead of individual outcome. In such cases, further analysis involving opinion dynamics may be required and this motivates future work. Single-prioritization voting systems can also be replaced with other game theoretic multi-voting systems.

In conclusion, we have shown that paradoxical games based on capital-dependent and history-dependent Parrondo's games can exhibit counter-intuitive phenomenon in large ensembles. Firstly, a plurality voting system is based on selfish voting and becomes inefficient when used for large ensemble of voters. Instead, by removing the spoiler option, we have shown that plurality vote now becomes majority vote, which can lead to gain in average capital per voter. This in itself is counter-intuitive for two reasons: (1) Dinís and Parrondo [27] have shown that 2-option controlled collective games are inefficient, for large ensemble of voters, but we have presented an example where the 2 options are between games B and C. (2) By replacing a low-risk game, i.e. game A, for a higher risk game, i.e. game C, plurality voting led to positive average capital per player. Secondly, the rank choice voting system is effective for large group voting, arriving at single prioritization. For the same number of voters in the ensemble, rank choice voting always leads to a higher average capital per player than plurality voting. Furthermore, we have shown that rank choice voting is not susceptible to the spoiler option. However, it may not be beneficial to options that have low risks, like game A. The centre squeeze phenomenon is observed, as game A is usually not chosen as the game of choice under rank choice voting. Thus, the outcome of a 3-option or 2-option controlled collective game does not vary significantly. Lastly, for the approval voting system, we have shown that by assigning probabilities of approval to each choice, the outcome leads to the emergence of the Volunteer's dilemma. In the case of approval voting, despite the majority of the domain of r_2 and $r_{3|2}$ leading to positive average capital per voter, a small change in the probabilities from the "Goldilocks" zone can significantly decrease the average capital gained.

Availability of data and materials All data generated or analysed during this study are included in this paper.

Declaration

Conflict of interest The authors declare that they have no conflict of interest.

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