Parrondo paradoxical walk using four-sided quantum coins

Joel Weijia Lai
Science, Mathematics and Technology, Singapore University of Technology and Design (SUTD), 8 Somapah Road, Singapore 487372

Jean Ren Adriel Tan,* Huiyi Lu,* and Zi Rou Yap*
National University of Singapore High School of Mathematics and Science, 20 Clementi Ave 1, Singapore 129957
and Science, Mathematics and Technology, Singapore University of Technology and Design (SUTD), 8 Somapah Road, Singapore 487372

Kang Hao Cheong†
Science, Mathematics and Technology, Singapore University of Technology and Design (SUTD), 8 Somapah Road, Singapore 487372
and SUTD–Massachusetts Institute of Technology International Design Centre, 8 Somapah Road, Singapore 487372

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Two losing games can be played in a certain manner to produce a winning outcome—a phenomenon known as Parrondo's paradox. Of particular interest is the emergence of quantum game theory and the attempt to model known Parrondo's games through quantum computation notation. In this article, we investigate whether flipping four-sided quantum coins will result in the emergence of Parrondo's paradox. We discover that by playing two losing games A and B in a sequential order, a winning scenario can be derived. Furthermore, four-sided quantum coin is the first instance where the ratcheting effect from the classical Parrondo's game is necessary. Crucially, our study is designed with quantum protocols as its basis and does not have a direct classical counterpart.

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I. INTRODUCTION

Parrondo's paradox refers to the counterintuitive phenomenon whereby a winning strategy is obtained by combining two individually losing games in a certain manner [1]. Inspired by the flashing Brownian ratchets, it involves switching a spatially periodic potential on and off to produce a net drift of Brownian particles, despite no large-scale gradient in the potential. Since then, variants of Parrondo's games, such as the history-dependent [2] and cooperative games [3], have inspired numerous works on their mathematical properties and applications [4–7]. Parrondo's paradox has attracted much attention since its conception. With its theoretical advancements [5,6,8,9], Parrondo's paradox has enriched our understanding of a wide range of physical phenomena across various fields. In physics, it has been used to understand the dynamics in diffusive flow [10,11], and in biophysics, it has been useful in the modeling of molecular motors [12]. There are also applications in engineering optimization [13,14]. In systems chemistry, greater product yields are predicted to be achievable by exploiting the Parrondo effect through nonequilibrium cycling [15]. The paradox has also found numerous applications across ecology and evolutionary biology [16–24], as well as social dynamics [25–29]. Recently, it has been used in the study of information exchange in a social network on investment [30].

Of particular interest is the emergence of quantum game theory and the attempt to model known Parrondo's games through quantum information and computation notation, summarized in this review article [31]. The recent rise in quantum game theory and the representation of biased coins as qubits have led to the development of quantum Parrondo's games. In quantum games, "noise" is inherently built into the system with the probabilistic nature of superposition of qubit states in quantum mechanics.

While the classical capital-dependent Parrondo's paradox can only be realized by three 2-sided biased coins, it is possible to realize the same phenomenon in quantum Parrondo's paradox with two 2-sided fair quantum coins. Previous research has shown that Parrondo's paradox can be observed within quantum walks through the use of two 2-sided quantum coins and one 3-sided quantum coin [32,33]. Recently, a similar simulation analysis was performed by executing an identical methodology comprising two 2-sided fair quantum coins on a quantum random walk [34]. Their results reveal that the presence of ratcheting effect is not necessary for two 2-sided quantum coins to actualize the paradox. The underlying nature of quantum superposition naturally creates the randomized ratcheting effect required for the emergence of Parrondo's paradox.

There also remains a critical gap in the generalizability of Parrondo's paradox to 2^n-sided quantum coins. The present work seeks to narrow that gap by exploring the paradoxical dynamics of tossing 4-sided ($n = 2$) quantum coins. In doing so, it reveals the dynamics not observed in any of the 2-sided or 3-sided quantum coin toss games. In particular, the tossing of 4-sided quantum coins is the first instance of the

*These authors contributed equally to this work.
†Corresponding author: kanghao_cheong@sutd.edu.sg

ratcheting effect observed in classical games. Beyond which, it is still very much left unexplored, especially with regard to the applications of Parrondo's paradox through quantum coin toss in quantum walks.

In Sec. II, we introduce the formalism of a quantum coin toss in the form of operators and set up the initial states that lead to the emergence of the paradox. Then, we present and discuss our results in Sec. III before concluding with the prospects of this work in Sec. IV.

II. QUANTUM COIN TOSS IN A RANDOM WALK: METHODOLOGY

The Hilbert space that the quantum coin toss in a random walk operates in, is $\mathcal{H}_4 \otimes \mathcal{H}_p$, where $\mathcal{H}_4$ is the 4-sided coin space and $\mathcal{H}_p$ the position space. The dynamics in the position space is coupled to the one in the coin space; however, the converse is not true. Each step of a quantum random walk comprises two transformations. The first transformation corresponds to the flipping of a quantum coin (the coin operator $\hat{\mathcal{C}}$), and the second transformation (a translation operator $\hat{\mathcal{S}}$) corresponds to the result of the first transformation. A quantum coin is tossed to decide the number of discrete steps, s_n, to take with reference to the result of the quantum coin flip. A general 4-sided quantum coin is an arbitrary superposition of four states

$$|c\rangle = \sum_{n=0}^{3} a_n |n\rangle_c, \text{ where } \sum_n |a_n|^2 = 1, \tag{1}$$

with basis

$$|0\rangle_c = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad |1\rangle_c = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad |2\rangle_c = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad |3\rangle_c = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}. \tag{2}$$

The transformation given by the coin operator $\hat{\mathcal{C}}$ takes the form of a 4×4 unitary matrix. The position of the walker on the line is represented as a superposition of the p possible states,

$$|x\rangle = \sum_k \alpha_k |k\rangle_p, \text{ where } \sum_k |\alpha_k|^2 = 1, \tag{3}$$

with the translation operator $\hat{\mathcal{S}}$ taking the following form:

$$\hat{\mathcal{S}} = \sum_{n=0}^{3} \left[|n\rangle_c \langle n| \otimes \sum_k |k + s_n\rangle_p \langle k| \right], \tag{4}$$

where s_n is the number of steps taken by the quantum walker in a quantum walk. Thus, the entire transformation is unitary and combines the coin operator and translation operator, given by

$$\hat{\mathcal{U}} = \hat{\mathcal{S}}(\hat{\mathcal{C}} \otimes \hat{\mathcal{I}}_p), \tag{5}$$

where $\hat{\mathcal{I}}_p$ is the identity operator of size $p \times p$. The initial state of the system is $|\psi\rangle_0 = |c\rangle_0 \otimes |x\rangle_0$, and N steps are taken by applying the unitary operator N times,

$$|\psi\rangle_N = \hat{\mathcal{U}}^N |\psi\rangle_0. \tag{6}$$

The choice of operators and initial state are important in revealing the Parrondo effect. The *coin operator* takes the form of the following Hadamard-like matrices:

$$\hat{\mathcal{C}}_A = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}, \text{ and}$$

$$\hat{\mathcal{C}}_B = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & iAe^{i\beta} & -1 & -iAe^{i\beta}\\ 1 & -1 & 1 & -1\\ 1 & -iAe^{i\beta} & -1 & iAe^{i\beta}\end{bmatrix}, \text{ with } \beta \in [0, \pi), \tag{7}$$

where β plays the role of varying the phase of the state. If $\beta \neq \{0, \pi\}$, then $\hat{\mathcal{C}}_B$ must be normalized with an appropriate constant A, otherwise, $A = 1$.

Setup. For this article, in order to make the coin toss fair such that each basis state has equal weight for each toss, we set $\beta = 0$. The *translation operator* exhibits the ratcheting effect present in classical Parrondo's games.

$$\hat{\mathcal{S}} = |0\rangle_c\langle 0| \otimes \sum_k |k-1\rangle_p\langle k| + |1\rangle_c\langle 1| \otimes \sum_k |k\rangle_p\langle k| + |2\rangle_c\langle 2| \otimes \sum_k |k+1\rangle_p\langle k| + |3\rangle_c\langle 3| \otimes \sum_k |k+1\rangle_p\langle k|. \tag{8}$$

This ratcheting effect is seen from the boost in the positive direction for states $|2\rangle$ and $|3\rangle$, as opposed to a single boost in the negative direction for the state $|0\rangle$. This ratcheting effect is necessary for the Parrondo effect to be observed, which is not necessary in the analogous 2-sided quantum Parrondo's game. Finally, the chosen *initial state* is

$$|\psi\rangle_0 = |0\rangle_c \otimes |0\rangle_p. \tag{9}$$

Numerical example. To demonstrate the mechanics behind the methodology, we illustrate with a numerical example below. Consider the state $|x, s\rangle$ from the Hilbert space $\mathcal{H}_4 \otimes \mathcal{H}_p$, where x indicates the position of the state and $s \in \{0, 1, 2, 3\}$ determines the basis of the transformation and the corresponding boost. The coin operator $\hat{\mathcal{C}}_A$ acts on the four possible states in the following manner:

$$\begin{aligned}
\hat{\mathcal{C}}_A|x,0\rangle &= \tfrac{1}{4}|x,0\rangle + \tfrac{1}{4}|x,1\rangle + \tfrac{1}{4}|x,2\rangle + \tfrac{1}{4}|x,3\rangle,\\
\hat{\mathcal{C}}_A|x,1\rangle &= \tfrac{1}{4}|x,0\rangle - \tfrac{1}{4}|x,1\rangle + \tfrac{1}{4}|x,2\rangle - \tfrac{1}{4}|x,3\rangle,\\
\hat{\mathcal{C}}_A|x,2\rangle &= \tfrac{1}{4}|x,0\rangle + \tfrac{1}{4}|x,1\rangle - \tfrac{1}{4}|x,2\rangle - \tfrac{1}{4}|x,3\rangle,\\
\hat{\mathcal{C}}_A|x,3\rangle &= \tfrac{1}{4}|x,0\rangle - \tfrac{1}{4}|x,1\rangle - \tfrac{1}{4}|x,2\rangle + \tfrac{1}{4}|x,3\rangle.
\end{aligned} \tag{10}$$

The translation operator in Eq. (8), then translates each state in the following manner:

$$\begin{aligned}
\hat{\mathcal{S}}|x,0\rangle &= |x-1,0\rangle,\\
\hat{\mathcal{S}}|x,1\rangle &= |x,1\rangle,\\
\hat{\mathcal{S}}|x,2\rangle &= |x+1,2\rangle,\\
\hat{\mathcal{S}}|x,3\rangle &= |x+1,3\rangle.
\end{aligned} \tag{11}$$

Hence, combining the rules given by Eqs. (10) and (11), we obtain the mechanics of a single quantum coin toss by

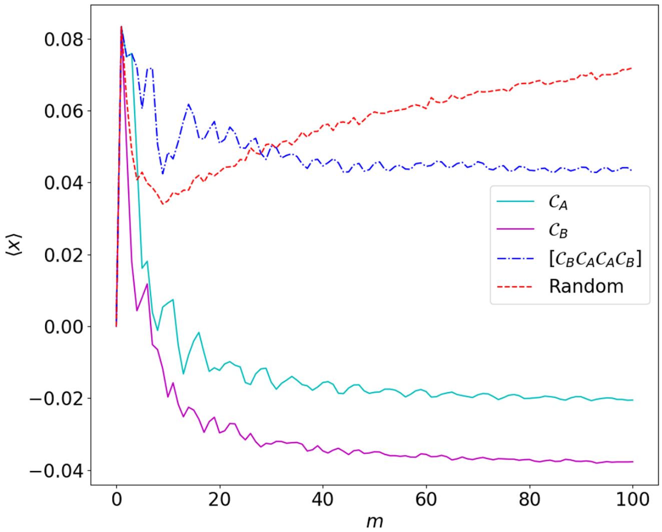

FIG. 1. The Parrondo effect exhibited by quantum coin toss operators given by Eqs. (7) and (8) and the initial state given by Eq. (9). The figure shows the mean position of tossing coins A, B, the random combination of both games, and the sequence [BAAB].

composing $\hat{\mathcal{S}}\hat{\mathcal{C}}_A$. That is, starting with state $|0,0\rangle$

$$\hat{\mathcal{S}}\hat{\mathcal{C}}_A|0,0\rangle = \hat{\mathcal{S}}\big[\tfrac{1}{4}|0,0\rangle + \tfrac{1}{4}|0,1\rangle + \tfrac{1}{4}|0,2\rangle + \tfrac{1}{4}|0,3\rangle\big]$$
$$= \tfrac{1}{4}|-1,0\rangle + \tfrac{1}{4}|0,1\rangle + \tfrac{1}{4}|1,2\rangle + \tfrac{1}{4}|1,3\rangle. \quad (12)$$

At this juncture, it is already evident that there is a probability of 0.5 to be in position $x = 1$ and a probability of 0.25 to be in position $x = 0$ and $x = -1$ each. A sequential quantum coin toss will cause these probabilities to evolve and the superposition principle may result in some states being annihilated or amplified. This is the intuition behind the results that we are going to report in the next section.

III. RESULTS AND DISCUSSION

The quantum coin toss operators given by Eqs. (7) and (8) and the initial state given by Eq. (9) were used as the preliminaries to a numerical simulation. Furthermore, four different simulations were performed for the quantum coin toss, corresponding to the combination of using operators $\hat{\mathcal{C}}_A$ and $\hat{\mathcal{C}}_B$. Two games were simulated using operator $\hat{\mathcal{C}}_A$ only and operator $\hat{\mathcal{C}}_B$ only. A third game was played by randomly tossing, with fair probability, both operators. The random nature of the third game takes an averaging over $n = 1000$ games, so that the mean outcome is obtained. Lastly, a fourth simulation was performed by sequentially tossing both coins with period [BAAB]. The results for $m = 100$ steps are shown in Fig. 1.

The simulation results for all other periodic sequences of length at least 4 are compiled in Table I. The table shows that under random switching and periodic sequential tossing, it is now possible for two negative-position quantum walks to be combined to give a positive-position quantum walk. We believe that this is not the only possible combination of operators, initial state, and sequence. Using the same coin operators, acting on initial state $|\psi\rangle_0 = |0\rangle_c \otimes |2\rangle_p$ also gives negative-position quantum walks that can combine to give a positive-position quantum walk for the periodic sequence [ABBA] for the translation operator,

$$\hat{\mathcal{S}} = |0\rangle_c\langle 0| \otimes \sum_k |k\rangle_p\langle k| + |1\rangle_c\langle 1| \otimes \sum_k |k+1\rangle_p\langle k| + |2\rangle_c\langle 2| \otimes \sum_k |k-1\rangle_p\langle k| + |3\rangle_c\langle 3| \otimes \sum_k |k+1\rangle_p\langle k|. \quad (13)$$

This forms an important aspect of the work here, which is to classify the domain for which Parrondo's paradox is observed, with different initial states of the form $|\psi\rangle_0 = |0\rangle_c \otimes |i\rangle_p$, for $i = 0, 1, 2, 3$.

It is important to note that the sequences are noncommutative; for example, the sequence [BAAB] gives the highest positive mean position, and its cyclic variants [AABB], [ABBA], and [BBAA] all give different mean positions. By extension, our simulation results allow us to conjecture that if we succeed in finding a translation operator $\hat{\mathcal{S}}$ for some initial state $|\psi\rangle_0$ that exhibits quantum Parrondo's paradox for one periodic sequence, then any periodic sequence for that distinct translation operator and initial state will also exhibit Parrondo's paradox. This significantly reduces the computational time required to determine whether there exists a periodic sequence for a given $\hat{\mathcal{S}}$ and $|\psi\rangle_0$. This conjecture is observed in Table I, where all periodic sequences up to length 4 have positive mean positions for the distinct initial state and translation operator.

Furthermore, we observe from Fig. 1 that the position of the individual quantum coins and sequential coin toss stabilize

TABLE I. The average position after $m = 100$ steps; a positive average position suggests a net "win" in a quantum coin toss in a random walk. This table only shows sequences of periods up to length 4, all of which produce winning outcomes.

Game	$\langle x\rangle$	Game	$\langle x\rangle$	Game	$\langle x\rangle$
Coin A	−0.02052009	[ABA]	0.00950983	[ABBA]	0.00831107
Coin B	−0.03769104	[BAA]	0.02709054	[BAAB]	0.04310835
Random	0.07191560	[ABBB]	0.01019300	[BABA]	0.02066321
[AB]	0.01949955	[BABB]	0.03170744	[BBAA]	0.02319456
[BA]	0.02066321	[BBAB]	0.02964090	[AAAB]	0.02767047
[ABB]	0.00335323	[BBBA]	0.01140707	[AABA]	0.02666026
[BAB]	0.04039599	[AABB]	0.02397298	[ABAA]	0.02444058
[BBA]	0.00360020	[ABAB]	0.01949955	[BAAA]	0.02839085
[AAB]	0.02941601				

at $m = 100$, while the random coin toss continues to increase. This has been computationally verified for up to $m = 200$. As of now, there is no affirmative method to prove this result analytically, hence this motivates future work.

Lastly, we would like to highlight that such setup for the quantum coin toss in a random walk has not been previously observed in coins with 2 or 3 sides. In the case of a 4-sided quantum coin toss, Parrondo's paradox emerges due to the underlying presence of the ratcheting effect caused by two factors. Firstly, it arises from the superposition principle. Certain states are annihilated and others amplified during the process of quantum coin flipping. This cannot be the only reason because without the boost, Parrondo's paradox cannot be achieved with the given parameters from Eqs. (7) and (9) alone. Thus, a boost is necessary for the paradox to emerge. Secondly, as mentioned, a boost from the translation operator also provides the ratcheting effect. Again, the translation operator alone cannot be the only contributing factor to the emergence of Parrondo's paradox, because both coins A and B are subjected to the same translation operator. It is worth noting that the first condition has been sufficient in observing the paradox for the case involving 2-sided and 3-sided coins.

IV. CONCLUSION

The increased number of sides of the quantum coin gives rise to more complicated superposition states. Thus, the increased complexity necessitates the ratcheting effect in Eq. (8), as first observed in the classical capital-dependent game and later in other Parrondo's games. The 4-sided quantum coin is the first occurrence of the need for this ratcheting effect provided by a boost in states, not observed previously in 2-sided and 3-sided coins.

Quantum computation has been a keen area of interest for the past few years, especially with its applications in quantum game theory [35,36]. Quantum walks have proven to be useful in simulating physical systems, as well as in developing new and better quantum algorithms. As quantum walks spread quadratically faster than classical random walks, an algorithm implemented on a quantum walk will require much less time to complete than one implemented on a classical walk—another step closer to quantum supremacy [37]. The successful implementation of Parrondo's game on a quantum walk can provide insights into new quantum algorithms. Additionally, we would like to note that there is a possibility that we can always observe the Parrondo effect for any 2^n-sided quantum coin. The task is to find the correct operators, initial states, and sequence. As quantum information theory continues to develop, we hope that many of the open problems discussed in this article will eventually be solved analytically.

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