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REVIEW

Social dynamics and Parrondo's paradox: a narrative review

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Abstract How do group dynamics affect individuals within the group? How do individuals, in turn, affect group dynamics? As society comes together, individuals affect the group dynamics and vice versa. Social dynamics look at group dynamics, its effect on individuals, conformity, leadership, networks, and more. In the past two decades, the game theoretic Parrondo's paradox has been used to model and explain the different aspects of social dynamics. Two losing games can be combined in a certain manner to give a winning outcome—this is known as Parrondo's paradox. In this review, the connections between Parrondo's paradox and social dynamics are discussed with emphasis on (i) cooperation and competition, (ii) resource redistribution and social welfare, and (iii) information flow and decision-making.

Keywords Social dynamics · Parrondo's paradox · Sociodynamics · Information flow · Game theory · Nonlinear analysis

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1 Introduction

One of the ways to study social dynamics is to investigate social changes in agent-based models that involve multiple interacting individuals. These individuals, with differing characteristics, rules of behaviour, and sources of information, collectively form a group. The interaction may lead to changes in behaviour of the group as a whole, which inadvertently shapes the behaviour of individuals in the group. The dynamics of such systems can be extremely complex due to their high dimensionality and inter-connectivity between individuals [1, 2]. Social dynamics is more than just summing up the individual characteristics of each group member. Instead, social dynamics involves a cyclical and reciprocal feedback loop which simultaneously impacts individuals and the connectivity between individuals. The study of social or group dynamics helps to improve group performance, communication, and consensus [3–5]. At the forefront of this research is the applicability of mathematical modelling to sociodynamical phenomena [6].

Since the time of von Neumann, game theory has been applied to a wide range of behavioural relations [7–9]. Game theory rigorously analyses the long-run behaviour of agent-based systems using concepts in Markov chains and equilibrium analysis [10–12]. Analysis of interactions in a group often reveals counter-intuitive results. For example, Braess's paradox states that adding one or more roads

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to an existing road network may end up slowing down overall traffic flow. This is because individuals wish to optimise their own utility rather than considering the group utility [13]. This is likened to finding a Nash equilibrium in a multi-player game, where the Nash equilibrium does not necessarily give the best possible result for any player. Another counter-intuitive and nontrivial result of game theory is Parrondo's paradox.

Two losing games can be combined in a certain manner to give a winning outcome—this is known as Parrondo's paradox [14, 15]. It has been used to explain various phenomena in computer science [16], with its theoretical framework [17–22] expanding into quantum game theory [23–26], physical chemistry [27], dynamical systems [28–30], and engineering [31, 32]. Furthermore, of importance, where Parrondo's paradox has made a breakthrough, is the observed phenomenon in biology where there are interacting agents in an ecosystem with the environment and living space, giving rise to primitive modelling of aspects of population dynamics [33–43]. These models go beyond social interactions to take into account the effect of environments on populations. The modelling of interaction between multi-agents, in social dynamics, is of greater complexity as there is greater connectivity between the agents, which makes the fusion of Parrondo's paradox and social dynamics an important and emerging field of nonlinear dynamics.

It has been more than two decades since the formalism of Parrondo's paradox. This review article hopes to consolidate the works in this field pertaining to social dynamics—the coming together of two important topics from the game theory and social sciences to explain how two "losing" utilities can lead to a "winning" utility in society. To achieve the aims of this work, our article is organised as follows: the next section will present a discussion on the three main types of Parrondo's games (Sect. 2). This will be followed by a review of the social dynamics context in which the study is focused (Sect. 3). According to Comte [44], there are three types of social progress: (i) moral progress, (ii) physical progress, and (iii) intellectual progress. Motivated by these broad themes, our review here discusses the existing literature, paying special attention to (i) cooperation and competition, (ii) resource redistribution and social welfare, and (iii) information flow and decision-making, with the view to evaluate the implications and future direction of this amalgamated subject. This review is important as it provides a different perspective through which one can understand societal ideas of redistribution, cooperation, voting, performance, and resource growth to bring about "winning" outcomes in a social group.

2 Parrondo's paradox

There are three broad classes of Parrondo's games—capital-dependent, history-dependent, and cooperative Parrondo's games.

2.1 Capital-dependent Parrondo's games

In capital-dependent Parrondo's games, a player starts with an initial capital $C(t)$ and plays one of the two losing games [14, 45]. At each time step t, the player chooses to play either game A or B according to some rule. The player's capital increases by 1 for each win, that is $C(t+1) = C(t) + 1$; otherwise, the player's capital decreases by 1, that is $C(t+1) = C(t) - 1$. Game A involves the toss of a single-biased coin with winning probability $p_0 = \frac{1}{2} - \varepsilon$. In Game B, we first determine whether current capital is a multiple of some integer. That is, coin c_1 with a winning probability of p_1 is tossed if the capital is divisible by integer M; otherwise, coin c_2 is used with winning probability p_2. By choosing p_1 and p_2, one is able to show that playing games A and B individually over time gives a net loss in capital, but playing a combination of games A and B randomly, denoted by $[\mathrm{A}+\mathrm{B}]$, results in a gain in capital. The choice of probabilities for game B that satisfies this condition is $p_1 = \frac{1}{10} - \varepsilon$, $p_2 = \frac{3}{4} - \varepsilon$ and $\varepsilon = 0.005$. This is summarised in Fig. 1, and results are plotted in Fig. 2.

It is instructive to introduce the representation of Parrondo's games in the form of a Markov chain as it is often used to analyse and classify equilibrium or detailed balance solutions. In the capital-dependent Parrondo's games, we have M states, each state corresponding to $i \equiv C \bmod M$, where $i \in \{0, 1, \ldots, M-1\}$. We can illustrate the transition from one state to another via Fig. 3

In both Markov chains, the clockwise direction is "winning", while the counter-clockwise direction is

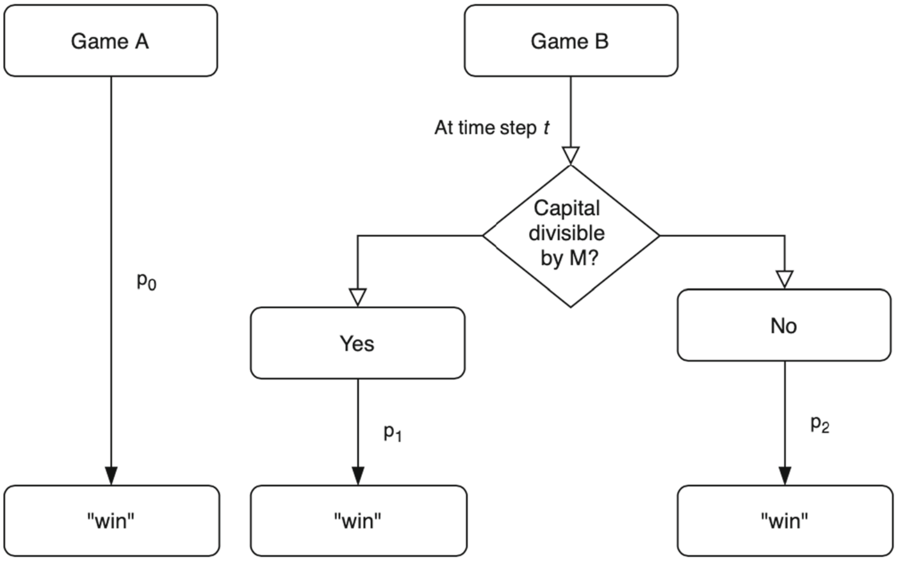


Fig. 1 Capital-dependent Parrondo's games involve tossing three biased coins as described in Ref. [14]. The probabilities for the losing condition are $1-p_i$ for the respective branches. This will be the convention adopted for the rest of this article

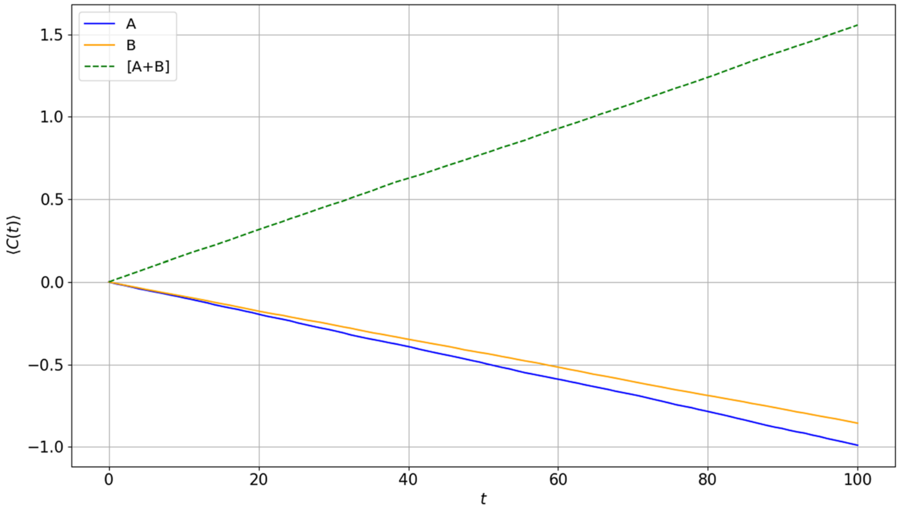


Fig. 2 Average capital $\langle C(t)\rangle$ against time. The following parameters were chosen such that Parrondo effect is observed. The capital is averaged over $n = 10^6$ simulations for $t = 100$ time steps for the following values of p: $p_0 = 0.5-\varepsilon$, $p_1 = 0.1-\varepsilon$ and $p_2 = 0.75-\varepsilon$, setting $\varepsilon = 0.005$

"losing". It is easy to check that game A is a losing game if

$$1-p_0 > p_0 \Rightarrow \frac{1-p_0}{p_0} > 1. \quad (1)$$

For game B, it is losing if

$$(1-p_1)(1-p_2)^{M-1} > p_1 p_2^{M-1} \Rightarrow \frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}} > 1. \quad (2)$$

For the Parrondo's games, we can mix game A and B such that the combined game C can be written as $C = \gamma A + (1-\gamma)B$, where $0 \le \gamma \le 1$. If $\gamma = 1$, then only game A is played, and if $\gamma = 0$, then only game B is played. In most literature, both games A and B are played with the same probability for the capital-dependent Parrondo's games, so $\gamma = \frac{1}{2}$. Then at the stationary distribution, Parrondo's paradox occurs if

$$\frac{(1-q_1)(1-q_2)^{M-1}}{q_1 q_2^{M-1}} < 1, \quad (3)$$

where $q_1 = \gamma p_0 + (1-\gamma)p_1$ and $q_2 = \gamma p_0 + (1-\gamma)p_2$. The probabilities p_i for the capital-dependent Parrondo's games, $M = 3$ and $\gamma = \frac{1}{2}$, satisfy this inequality.

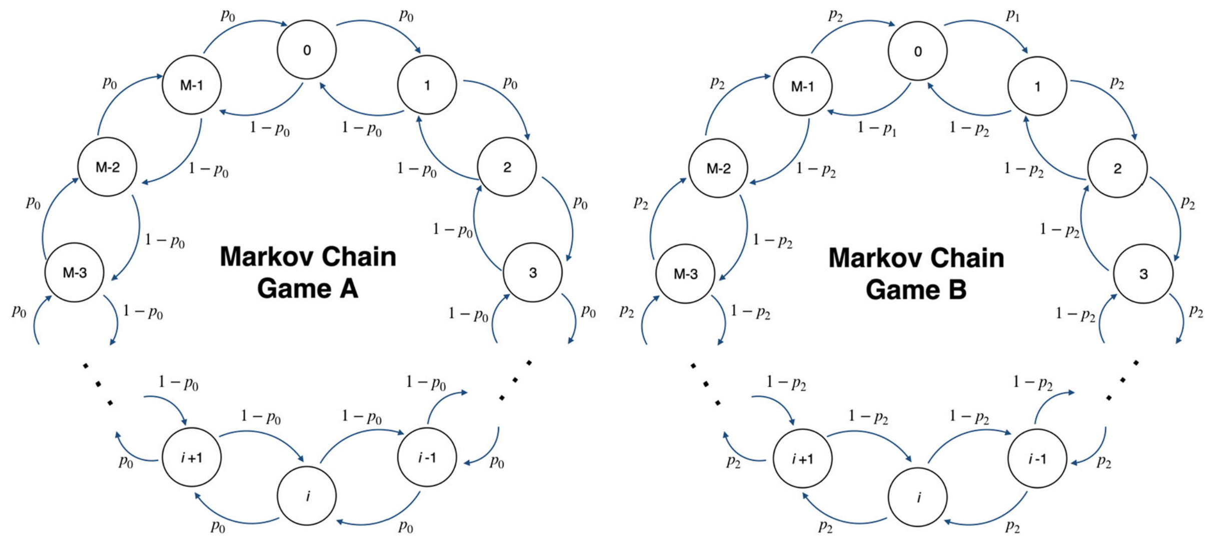


Fig. 3 The states i representing $i \equiv C \bmod M$. (Left) Markov Chain for game A: probability of winning is p_0 and probability of losing is $1-p_0$. (Right) Markov Chain for game B: If the capital is a multiple of M, then the probability of winning is p_1; if it is not a multiple of M, then the probability of winning is p_2. Image adapted from Ref. [46]

2.2 History-dependent Parrondo's games

The history-dependent Parrondo's games build on the concept of capital-dependent games by keeping game A to be similar as before. However, game B is now independent of the capital—we will call it game B$'$ so as not to confuse with the earlier game B. In game B$'$, there are four biased coins and the coin to be played depends on the *history* of the outcomes from previous games [47, 48]. At each time step t, the player chooses to play either game A or B$'$ according to some rule. When the player wins the selected game, the player's outcome is labelled as "win" and $C(t+1) = C(t) + 1$; otherwise, the outcome is labelled as "lose" and $C(t+1) = C(t) - 1$.

The rules for game A are similar to the capital-dependent game A. Game B$'$ is defined according to the history of the past two time steps. The probability of winning at time t is given by:

- p_1, if player "lose" at both $t-1$ and $t-2$,
- p_2, if player "lose" at $t-1$ and "win" at $t-2$,
- p_3, if player "win" at $t-1$ and "lose" at $t-2$,
- p_4, if player "win" at both $t-1$ and $t-2$.

The dynamics of the game is described in Fig. 4. Playing games A and B$'$ individually will produce a net loss over time, but playing a combination of games A and B$'$ can result in an overall win. As an illustration, the history-dependent Parrondo's games use biased dice with winning probabilities $p_1 = \frac{9}{10} - \varepsilon$, $p_2 = p_3 = \frac{1}{4} - \varepsilon$ and $p_4 = \frac{7}{10} - \varepsilon$, with $\varepsilon = 0.005$. The outcome of this simulation is shown in Fig. 5.

2.3 Cooperative Parrondo's games

The application of Parrondo's games to social dynamics mainly hinges on cooperative Parrondo's games [49]. Cooperative Parrondo's games are structured differently and, unlike their counterparts in capital-dependent and history-dependent games, these games are played by an ensemble of agents following certain social rules. Much like social interaction, this set of "social rules" is determined by the interaction between agents.

In cooperative Parrondo's games, consider N players each with individual capital $C_i(t)$, $i = 1, \ldots, N$. This capital evolves by a combination of two games, A and B$''$. In the version of the game introduced by Toral [49], at a time step t, a player i is randomly chosen among the N players. Player i then chooses to play either game A or B$''$ according to some rule. When player i wins the selected game, the particular player is labelled as "win" and $C_i(t+1) = C_i(t) + 1$; otherwise, the outcome is labelled as "lose" and $C_i(t+1) = C_i(t) - 1$. Player i's label does not change until the player is

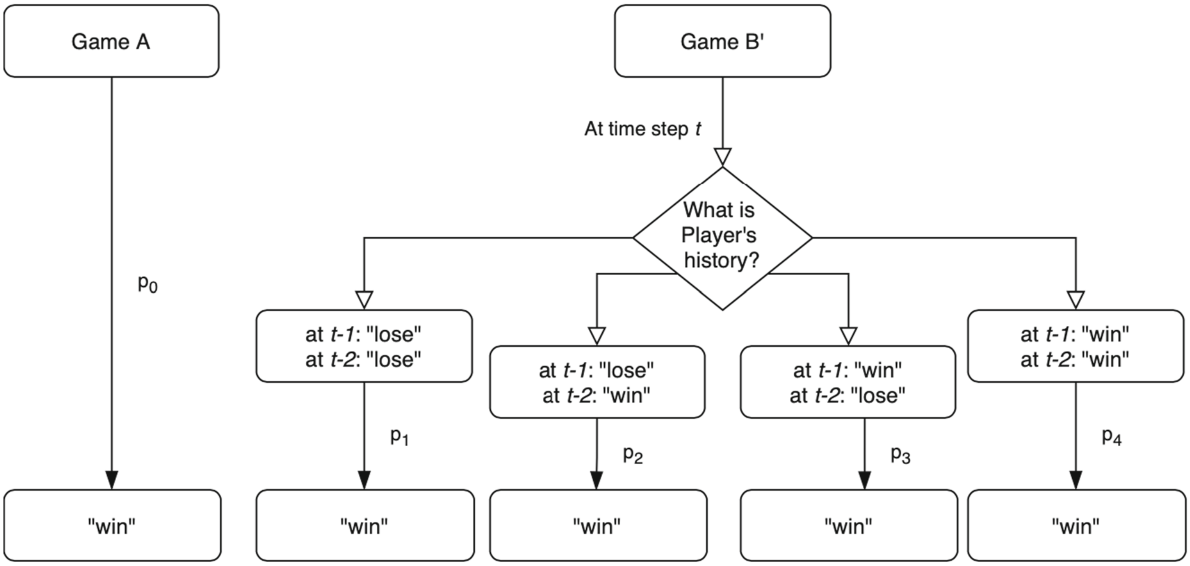


Fig. 4 Construction of the history-dependent games, where game B$'$ has four possible historic outcomes. Depending on the four historic outcomes of the game, the probability of winning is p_i, $i = \{1, 2, 3, 4\}$. Image adapted from Ref. [47]

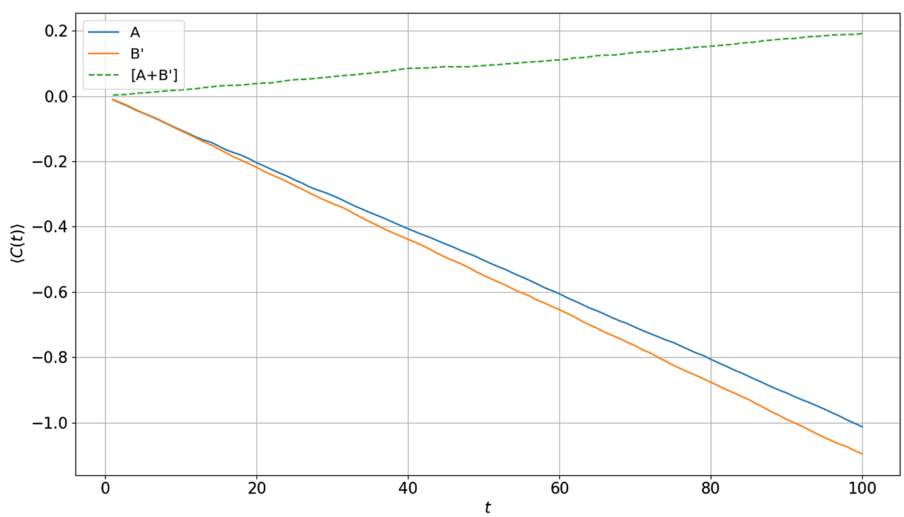


Fig. 5 Average capital $\langle C(t) \rangle$ against time. The following parameters were chosen such that Parrondo effect is observed. The capital is averaged over $n = 10^6$ simulations for $t = 100$ time steps for the following values of p: $p_0 = 0.5 - \varepsilon$, $p_1 = 0.9 - \varepsilon$, $p_2 = p_3 = 0.25 - \varepsilon$ and $p_4 = 0.7 - \varepsilon$, setting $\varepsilon = 0.005$

chosen again to play a game. The combination of games is considered a winning game if the average total capital, $\langle C(t) \rangle$, increases with time, where

$$C(t) = \sum_{i=1}^{N} C_i(t). \tag{4}$$

The dynamics of the cooperative Parrondo's games is illustrated in Fig. 6.

The rules for game A are similar to the ones described in Sect. 2.1. Game B$''$ (not to be confused with earlier versions) is defined according to the state of the two neighbouring players $i+1$ and $i-1$ (assume periodic boundary conditions). The probability of winning at time t is given by:

- p_1, if both players at site $i-1$ and $i+1$ are labelled "lose",
- p_2, if the players at site $i-1$ is labelled "lose" and the player at $i+1$ is labelled "win",
- p_3, if the players at site $i-1$ is labelled "win" and the player at $i+1$ is labelled "lose",
- p_4, if both players at site $i-1$ and $i+1$ are labelled "win".

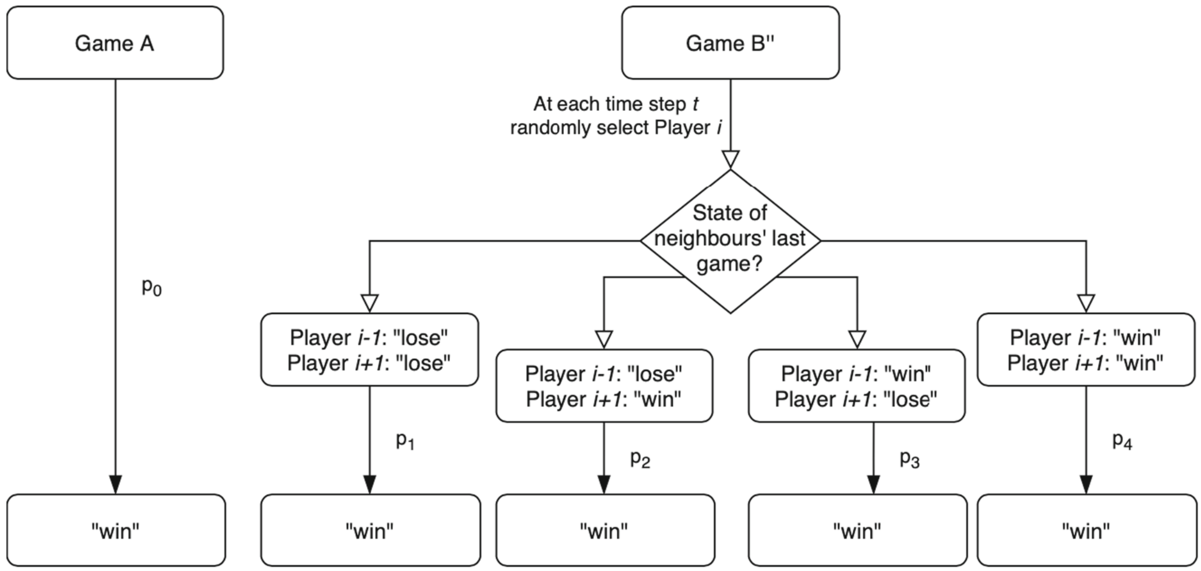


Fig. 6 Flowchart diagram for cooperative Parrondo's games, where game B$''$ has four possible neighbouring states. Depending on the four states of Player i's neighbours, the probability of winning is p_i, $i = \{1, 2, 3, 4\}$. Image adapted from Ref. [49]

The dynamics of choosing which game to play will determine the final outcome. A random game, denoted as $[\mathrm{A} + \mathrm{B}'']$, plays games A and B$''$ in random succession such that as $t \to \infty$, there is a probability 1/2 of choosing game A and probability 1/2 of choosing game B$''$ at each time step. The chosen parameters (leading to a winning outcome) are $p_0 = 0.5$, $p_1 < 0.5395$ or $p_1 > 0.98$, $p_2 = p_3 = 0.16$ and $p_4 = 0.7$. Players form a ring of N nodes (i.e. agent 0's neighbours are agent 1 and agent $N - 1$). The simulation results for a chosen set of parameters displaying Parrondo's paradox are shown in Fig. 7.

The cooperative game is the first multi-agent game to display Parrondo's paradox and lays the foundation for game theoretic Parrondo's games modelling of various aspects of social dynamics.

Since the introduction of cooperative Parrondo's games, the theoretical framework of these games has been extended and analytical derivations of the cooperative Parrondo's games have been carried out to identify the values of $p_j, j \in \{0, 1, \ldots, 4\}$ for population size N which gives rise to the Parrondo effect [50–58]. Modifications were made to include

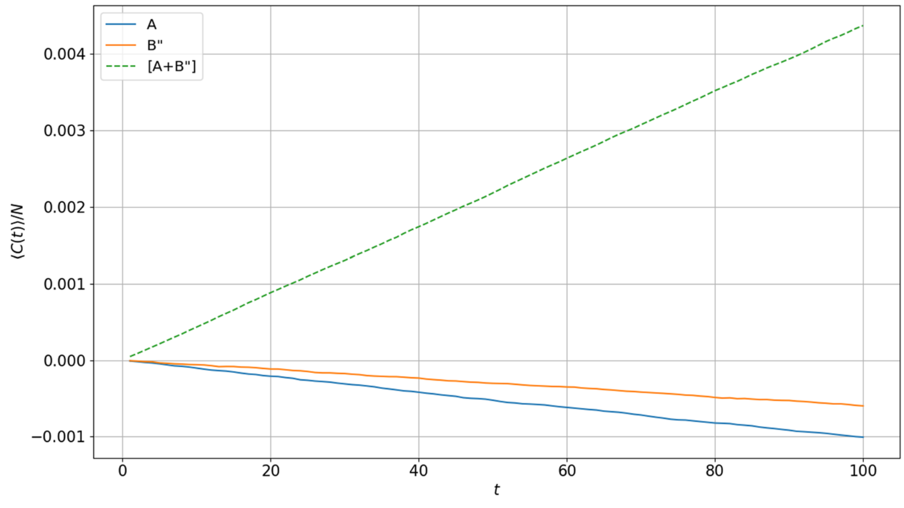


Fig. 7 Average capital per player $\langle C(t)\rangle/N$ against time. The following parameters were chosen such that Parrondo effect is observed. $N = 100$, averaged over $n = 10^7$ simulations for $t = 100$ time steps for the following values of p: $p_0 = 0.5 - \varepsilon$, $p_1 = 1.0$, $p_2 = p_3 = 0.16$ and $p_4 = 0.7$, setting $\varepsilon = 0.0005$

synchronous (all players play at the same time for each time step) cooperative Parrondo's games [59]. Following the same game rules, it is possible to arrange $N \times M$ players in a two-dimensional lattice (assume periodic boundary conditions) to achieve Parrondo effect for games played in an asynchronous (one player is chosen at every time step) [60] and a synchronous [61] manner. These theoretical advancements allow for more complex and realistic social dynamics to be explored.

3 Aspects of social dynamics and Parrondo's paradox

Parrondo's paradox, in the context of modelling social dynamics, is an agent-based model that involves a large (but finite) number of interacting agents with differing rules of behaviour or information. These behaviours and information are often simplified to remain analytically tractable. The dynamics of the interaction can be implemented as a mathematical function or algorithmic procedure [62, 63]. Most real-world social systems are complex due to the large number of connections between agents, and high dimensional due to the number of features describing behaviours and information. In all the presented work, as is common in social dynamics research, the agents act rationally. Their choices are also not entirely deterministic as it may be affected by internal factors such as errors in perception and idiosyncrasies in behaviour, as well as external factors such as random perturbations from the environment. In this section, we will review related work involving Parrondo's paradox on long-term behaviour in social dynamics. In modelling social dynamics, the "capital" introduced in Parrondo's games can represent many social aspects. For example, the most direct analogy of capital is money itself. It can also be regarded as certain benefits that flow from the trust, reciprocity, information, and cooperation associated with social networks.

3.1 Cooperation and competition

A key aspect of social dynamics is the study of how interaction within a group leads to cooperation and competition among its agents. These interactions are often modelled using a complex network that has a small-world topology. For the modelling to be realistic, there are local and global measures that have to be made so that predictions can be inferred on the group dynamics and its implications for the evolution of behaviours. There are several ways to define the emergence of cooperation or competition, and they are discussed in the work of Wang and Ye et al. [64–66].

In the work of Wang et al. [64], they have investigated agent's cooperation and competition (coopetition) behaviour, as well as the impact of heterogeneity of the degree distribution of network on coopetition. The agent-based Parrondo's games consist of a zero-sum game (game A) among agents and a negative sum game (game B) between agents and the environment. Considering a population of N agents, at each time step, a principal i is randomly chosen to play game A with probability p or game B with probability $1-p$. If game A is chosen, a receptor j is chosen, where j is connected to i, where $i \neq j$. Two networks were chosen—a fully connected network and the Barabási–Albert (BA) model scale-free network. In both networks, each agent is represented by a node. Agents that have means of interaction will be connected with an edge. In a fully connected network, all nodes are pair-wise connected. In the BA scale-free network, the nodes are connected based on a degree distribution that follows a power distribution [67]. The dynamics of the Parrondo's game is summarised in Fig. 8.

Game A is designed to have no impact on the total benefit of the population; however, it changes the distribution of benefit in the population. The dynamics of game A, in Fig. 8, are determined by the interaction relationship between principal i and receptor j. Four patterns, corresponding to interactions, are defined:

1. Competition pattern: the winning probabilities of the principal i and receptor j are both 0.5. When i wins, j pays one unit to i; otherwise, i pays one unit to j.
2. Cooperation pattern: the principal i pays one unit to the receptor j for free.
3. Harmony-based pattern: when capital $C_i(t) \geq C_j(t)$ i pays one unit to j; otherwise, when $C_i(t) < C_j(t)$, j pays one unit to i.
4. Poor-competition–rich-cooperation (PCRC) pattern: when capital $C_i(t) \geq C_0$, the initial capital, principal i will cooperate with receptor j; otherwise, it will compete.

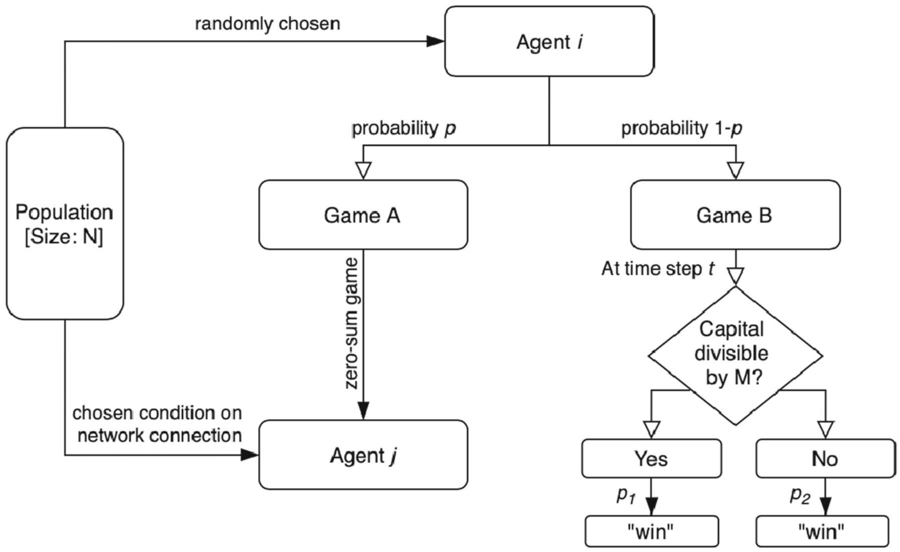


Fig. 8 Population Parrondo's games model on a network, modified from the capital-dependent Parrondo's games. Image adapted from Ref. [64]

Patterns (1)–(3) are "pure strategies". However, in reality, an individual cannot always have the same strategy and it will adjust its strategy accordingly. Thus, pattern (4), which is called "macrostrategy", is modelled according to the philosophy of self-cultivation, that is, "in success, commit oneself to the welfare of society; in distress, maintain one's own integrity".

Game B reflects the interaction mechanism between the population and the environment. It is specially designed to be a negative sum game similar to game B of the capital-dependent Parrondo's paradox introduced in Sect. 2.1. Game B is analogous to the ratcheting effect that natural environment has on biotic evolution. The two networks are compared by calculating the fitness index

$$d(t) = \frac{W(t)}{T}, \tag{5}$$

where $W(t) = C(t) - C_0$, $W(t)$ and $C(t)$ are the winnings and capital at time t, respectively. C_0 is the original capital, and T is the total time of the game with N, the population size. From this definition, the fitness of agent i and the average fitness of the population at time t are, respectively

$$d_i(t) = \frac{W_i(t)}{T}, \quad \text{and} \tag{6}$$

$$\bar{d}(t) = \frac{\left(\sum_{i=1}^{N} W_i(t)/N\right)}{T}. \tag{7}$$

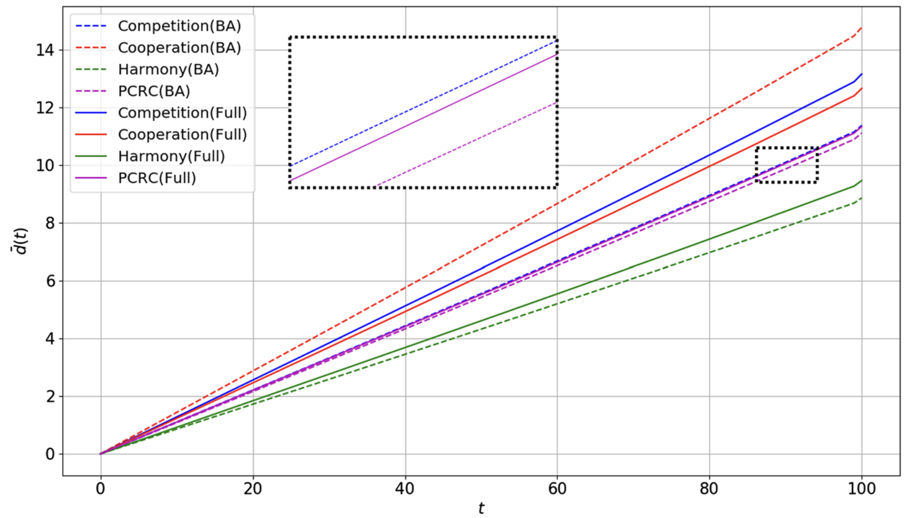
For both networks, $N = 500$, $T = 100$ and the probabilities are $p = 0.5$, $p_1 = 0.1 - \varepsilon$ and $p_2 = 0.75 - \varepsilon$, with $M = 3$ and $\varepsilon = 0.005$. Additionally, the BA network has an average degree of $\langle k \rangle = 3.984$ and clustering coefficient $G = 0.035652$. By calculating the average fitness index, the simulation results in Fig. 9 predict:

1. The BA network, which models many social networks, is conducive to cooperation.
2. Cooperation and competition in any forms are adaptive behaviours, and these behaviours could convert the losing games into a winning outcome. This implies that coopetition is in a successful evolutionary direction.

Further analytical and simulation results were later derived in [66], which also discusses the "Matthew effect" (associated with the effect of accumulated advantage), or simply known as "the rich get richer, while the poor get poorer". Furthermore, Ye et al. also investigated agent's coopetition in history-dependent Parrondo's games on networks [65]. Modifications were made to game A to investigate only the competition and cooperation patterns. Game B was substituted with game B′ to reflect the favourable and adverse impact of the environment on agents, modelled according to game B′ of the history-dependent game discussed in Sect. 2.2. The dynamics of the Parrondo's games are summarised in Fig. 10.

Game B′ is a negative sum game if

Fig. 9 Average fitness $\bar{d}(t)$ against time for both BA and fully connected networks. The following parameters were chosen such that Parrondo effect is observed. $N = 500$, averaged over $n = 100$ simulations for $t = 100$ time steps for the following values of p: $p = 0.5$, $p_1 = 0.1 - \varepsilon$ and $p_2 = 0.75 - \varepsilon$, with $M = 3$ and $\varepsilon = 0.005$



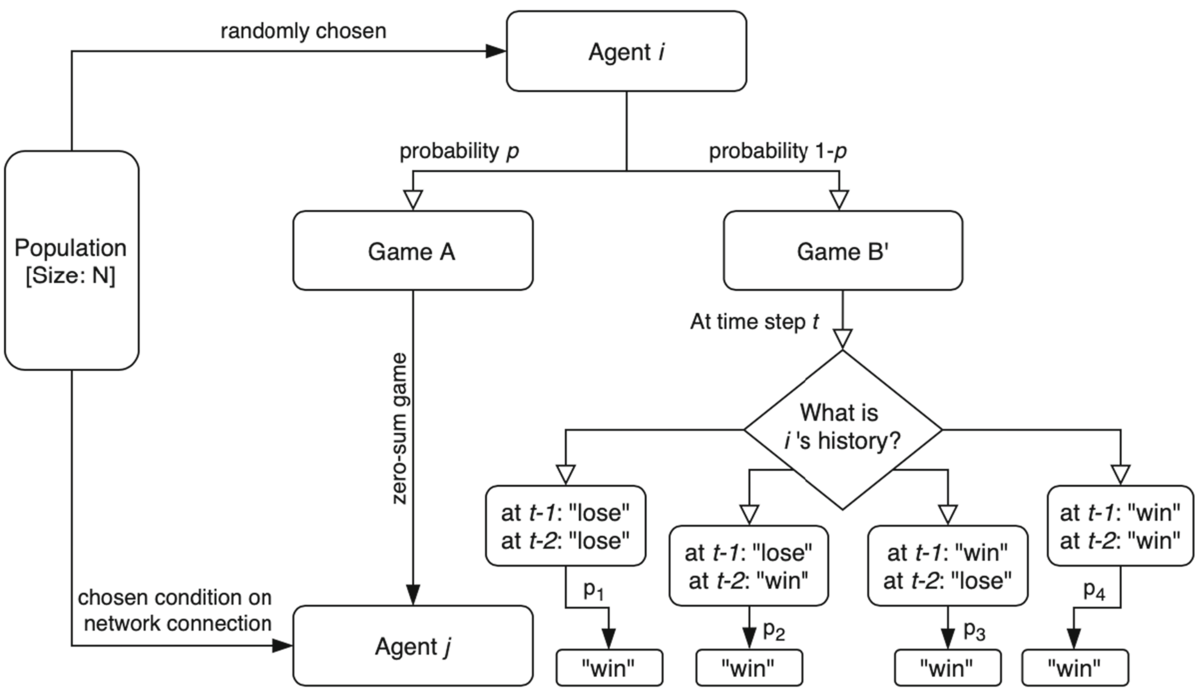


Fig. 10 Population Parrondo's games model on a network, modified from the history-dependent Parrondo's games. Image adapted from Ref. [65]

$$\frac{(1-p_3)(1-p_4)}{p_1 p_2} > 1 \quad (8)$$

The parameters are $p = 0.5, p_1 = 0.9, p_2 = p_3 = 0.21$ and $p_4 = 0.76$. Additionally, the BA network has an average degree of $\langle k \rangle = 3.984$ and clustering coefficient $G = 0.035650$. By making this modification to a history-dependent game and calculating the average fitness index, the simulation results in Fig. 11 predict:

1. Ye et al. arrived at the same conclusions as those from the population capital-dependent Parrondo's games.
2. In the BA network subjected to the cooperation pattern, agents with higher degree (i.e. connected

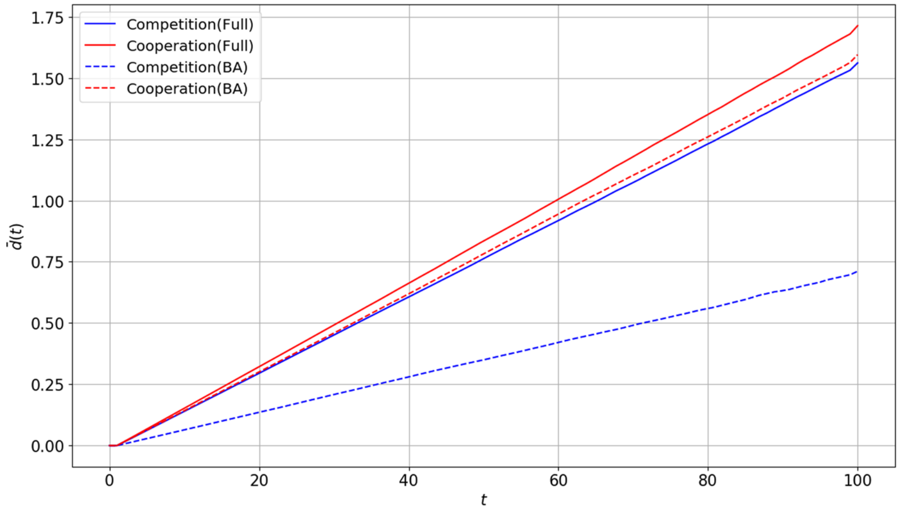


Fig. 11 Average fitness $\bar{d}(t)$ against time for both BA and fully connected networks. The following parameters were chosen such that Parrondo effect is observed. $N = 500$, averaged over $n = 100$ simulations for $t = 100$ time steps for the following values of p: $p = 0.5$, $p_1 = 0.9$, $p_2 = p_3 = 0.21$ and $p_4 = 0.76$

to more individuals) are observed to have higher fitness.

3. Coopetition behaviour (game A) makes agents in the network interact, which pushes flows of capital among agents and produces diversity of winning or losing states in the history. Therefore, coopetition behaviour results in the diversity which would promote the adaptation of the population.

These games show that social dynamics can be modelled effectively using Parrondo's games, and coopetition behaviour results in the diversity which will promote the adaptation of the population.

The dynamics of competition among agents is further studied by Arizmendi [68]. A competition model is applied to the context of competition between agents in a dating game. The matching problem involves two sets of agents matched pairwise. Each agent has a list of preferred partners from the other set. Agents that have each other ranked higher in the preference list have a higher probability to be accepted for a match. Arizmendi then asks the question, "Can the usual losers in the dating game achieve a better performance?" and shows that it is possible by considering a type of collective Parrondo's games. On the topic of competition, survival of the weakest is often observed as an emerging trend in systems. This is also discussed in the same vein in the works of Amengual et al. [69].

Any competition and survival of the weakest model can be thought of as a matching game. For generality, consider the dating game where there are N men and N women, who will interact for a time period T. Then, v_j^m is the "value" of woman j to every man, and similarly, v_i^w is the "value" of man i to every woman. These "values" remain constant with time. In each period, man i is chosen randomly. The expected man i's payoff of dating woman j is

$$\text{payoff}_{i,j}^m[t] = Q_{i,j}^m[t] \cdot p_{i,j}^m[t], \tag{9}$$

where $Q_{i,j}^m[t]$ is man i's estimate of the "value" of going out with woman j at time t and $p_{i,j}^m[t]$ is man i's estimate at time t of the probability that woman j will go out with him if he asks her out. The term, $p_{i,j}^m[t]$, models a man's decision that is based on prior beliefs and the number of benefits he has received. The expected "value" on a date and the probability that particular woman will accept his offer are taken account in Eq. 9. The expected woman j's payoff of dating man i is

$$\text{payoff}_{i,j}^w[t] = Q_{i,j}^w[t], \tag{10}$$

where $Q_{i,j}^w[t]$ is woman j's estimate of the "value" of going out with man i at time t. In this case, the probability term is not considered because woman j can only accept an offer if man i makes a proposal. Since the underlying v_j^m and v_i^w are constant,

$$Q_{i,j}^m[t] = \sum \left(v_j^m + v \right), \tag{11}$$

where the sum is made on the effective dates between i and j and v is noise drawn from a normal distribution. In the same way,

$$Q^{w}_{i,j}[t] = \sum \left(v^{w}_{i} + v\right). \quad (12)$$

Since the probability $p^{m}_{i,j}[t]$ changes with interaction and learning, it is updated according to

$$p^{m}_{i,j}[t] = (1-\eta)p^{m}_{i,j}[t-1] + \eta \frac{\text{offers accepted}_{i,j}[t-1]}{\text{offers made}_{i,j}[t-1]}, \quad (13)$$

where η is a constant learning parameter. The man's decision problem: the top ranked woman from the list of preferred partners of man i is selected to ask out for a date. The ranks in the preference list are determined according to the expected i's payoff of dating woman j according to Eq. 9. The man i acts in a greedy way, asking out woman j at the top of his preference list. The woman's decision problem: the rank of the woman's preference list is determined by the expected payoff according to Eq. 10. The decision depends on the two games.

(A) In game A, the women have two options to consider: exploration or exploitation. Exploitation is analogous to the greedy choice of maximising expected reward. For exploration, the women selects an action with lower expected payoff in the present in order to learn and increase future rewards. In game A, the exploration–exploitation trade-off depends on a probability distribution. The woman accepts the man's i offer to date with probability $p_A = 1/2$ (exploration) or she acts greedily and goes out with her best payoff$_w$ choice with probability $1 - p_A$.

(B) In game B, the choice of exploration or greedy behaviour is dependent on the collective state of all the men agents. A man is a "winner" if he receives his date by his last game; otherwise, he is a "loser". This exploration probability takes three possible values, determined by the number of winners w within the total number of players N, defined by

$$p_B = \begin{cases} p^{1}_{B} & \text{if } w > [2N/3] \\ p^{2}_{B} & \text{if } [N/3] < w \le [2N/3] \\ p^{3}_{B} & \text{if } w \le [N/3] \end{cases} \quad (14)$$

where $[\cdot]$ denotes the nearest integer. The woman accepts man i's offer to date with probability p_B (exploration) or she goes out with her payoff$_w$ choice with probability $1 - p_B$. The set of values $p^{k}_{B}, k \in \{1,2,3\}$ is chosen in order that the game remains fair and depend on the total number of players N, discussed in Ref. [51].

In the simulation performed by Arizmendi [68], it involves a group of $N = 4$ men and women. The learning rate is $\eta = 0.05$, and the probabilities are $p^{1}_{B} = 0.79, p^{2}_{B} = 0.65$ and $p^{3}_{B} = 0.15$. The noise signal is drawn from a normal distribution of standard deviation 0.5 and $v^{m}_{k} = v^{w}_{k} = N - k + 6$, where $1 \le k \le 4$. Arizmendi showed that losers benefit from the mixing of both games A and B. In the case where players are initially optimistic, but have their level of optimism declining over time, the loser acceptance increases by a factor of 10. The results are highly dependent on both the number N of players and the mixing probability of the games A and B. These results can generally apply to standard models of matching in economics or matching problems with partial information [70–72] to improve the outcome for the agent that is typically the weakest in a group in a matching game. The matching problem is an important problem in game theory, with applications in areas such as scheduling, planning, and network flows. More specifically, matching strategies are very useful in flow network algorithms such as the Ford–Fulkerson algorithm and the Edmonds–Karp algorithm. Parrondo's paradox can potentially be used to expand and improve the outcome of matching algorithms in these applications.

Roca et al. [73] discusses a wider problem in social dynamics relating to imitation dynamics. As an analogy to Parrondo's paradox, the authors managed to show that poor imitative behaviour due to local information can result in the promotion of cooperation, but these are often observed in the fast timescale behaviour.

3.2 Resource redistribution and social welfare

Resource redistribution is another key topic in social dynamics. It observes how the moral assessment of individual or collective decisions evolves in the light of how they affect distributions [74, 75]. However, while the number of independent factors affecting moral assessments may be difficult to quantify, models have been drawn up to quantitatively analyse its collective effect. The concept of redistribution has been invoked extensively in discussions of distributive justice in sociology; therefore, it is an important topic to study. In this section, we observe how modelling social groups through Parrondo's games can lead to increased resource or welfare in groups.

The original cooperative Parrondo's games were extended to feature wealth redistribution in another work by Toral [76]. As discussed in Sect. 2.3, it is possible for a multi-agent game to have a net increase in capital despite playing two losing games. In this extension, capital redistribution was considered for game A and the Parrondo effect is still observed.

In the first version, at each time step, an agent i is chosen at random. The chosen agent can play game A$'$ or game B with probability 1/2. Game B has the same mechanism as the game B in Sect. 2.1. If game A$'$ is chosen, then agent i gives away one unit of his capital to a randomly selected agent j. Game A$'$ is a losing game for agent i, but it is a zero-sum game across all players as it simply redistributes the capital. In the second version, the same game A$'$ is played, but game B is replaced by the game B$'$ as discussed in Sect. 2.2. In a third version, game A$'$ is replaced by another capital redistribution game A$''$. Agent i gives away one unit of its capital to any of its nearest neighbours, in a closed ring of N nodes, with probability proportional to their capital difference. These probabilities imply that the capital always goes from one agent to a neighbouring agent with a smaller capital and never otherwise. The results of playing any combination of these games are shown in Fig. 12. Notice that the combination of the games between A, A$'$ and A$''$ with B and B$'$ gives a positive capital—winning games, representing successful gain in average capital under redistribution.

These new versions of Parrondo's games involving ensemble of agents allow the redistribution of capital among the agents to be studied. Notice that the redistribution has no effect on the flow to total capital among the agents. By redistributing capital and combining it with other losing games, it can potentially increase the total capital made available.

Zappalà et al. [77] further developed the model designed by Toral by considering that all agents in a system may not always have the same strategy. Thus, in their extension, they included selfish and altruistic agents in the group. In the model, the population N is partitioned into selfish agents and altruistic agents. In the first version of the game, the strategy of selfish agents is to only play game B (the same game B as Sect. 2.1), while the strategy of altruistic agents is to randomly play two games, game A$'$ (the same game A$'$ as designed by Toral [76]) and game B. The dynamics of the game are illustrated in Fig. 13.

In a second version of the game, the strategies remain the same, except that the altruistic agents are selective as to who they give a unit of their capital to. Selective altruistic agents, when playing game A$''$, give a unit of capital if the recipient is also an altruistic agent. The dynamics of the game is illustrated in Fig. 14. When the population is allowed to change their behaviour, Zappalà et al. concluded that altruistic behaviour is discouraged because selfish agents tend to get richer, while altruistic agents get poorer. Instead, "selective altruism" reacts to altruistic behaviour according to reputation, or past information; this prevents selfish agents from taking advantage of altruistic agents. This also better reflects the real-world situation [78]. With a mechanism of reputation built into the model, altruistic agents can be aware of which agents are "trustworthy" so that they can overcome the negative effects of naive altruism. In fact, selectively altruistic agents obtain higher gains (in terms of capital) than selfish agents, eventually leading to selfish agents imitating them and in the population becoming altruistic. The works of Koh and Cheong [79, 80] and Ye et al. [81] provide further theoretical analyses of redistribution of wealth in networks by considering the redistribution games discussed.

3.3 Information flow and decision-making

In this section, we evaluate the use of Parrondo's games to model the flow of information leading to decision-making. A key function of social groups is information transfer. This requires communication amongst agents in the group [82]. In social or group dynamics, communication and information flow are often directed, as information can be personalised and targeted. This can thus result in social consensus or polarisation—herd behaviour. Agents change their decisions and beliefs on particular issues by interacting with other agents in the group and being influenced

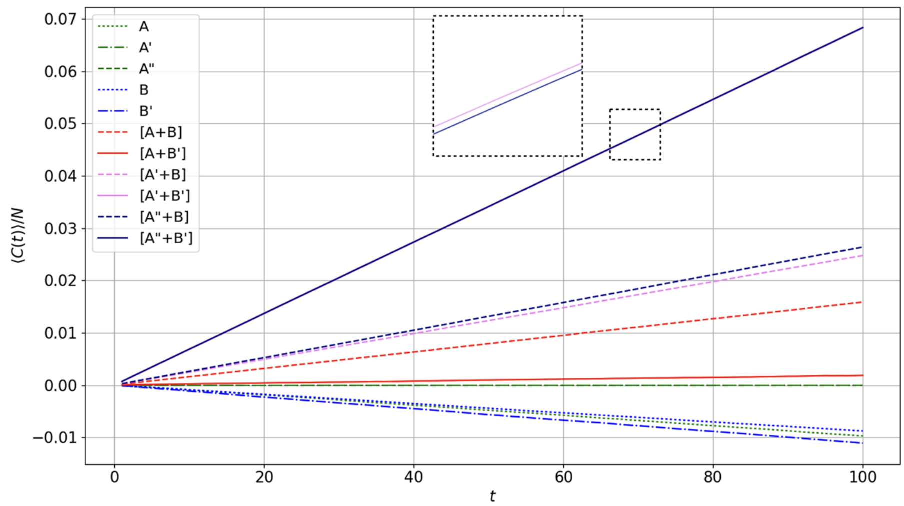


Fig. 12 Average capital per player $\langle C(t) \rangle / N$ against time for each game played independently (left) and the games played in a random combination (right). For the simulation, there are $N = 100$ agents, for a time duration of $t = 100$, averaged over $n = 10^6$ games. Note that since games A′ and A″ are both zero-sum games, their lines overlap at zero capital

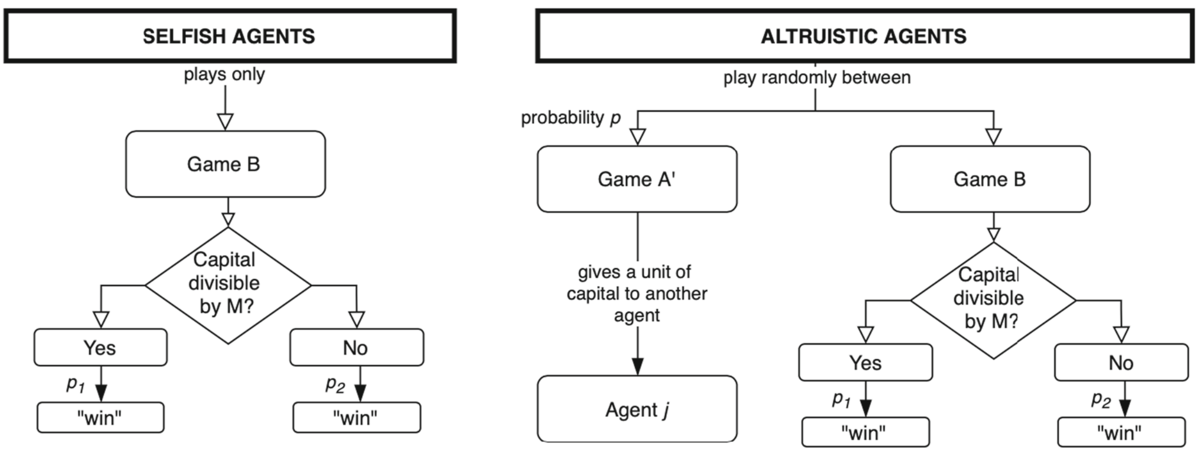


Fig. 13 Altruism-selfishness model played by a population *N* of fully connected agents. Selfish agents only play game B as they do not want to give away any capital. Altruistic agents consider two games A′ and B, where from time to time they consider giving away any capital. Image adapted from Ref. [77]

by external factors [83]. Changes in perception and decision can effect consensus or partisanship, which in turn can affect group action in social activities such as voting.

Early work on decision-making and voting using the Parrondo's model comes from Dinís and Parrondo et al. [84, 85]. This was later extended to collective voting by Xie et al. [86], and it follows the same development as Ref. [64]. Dinís and Parrondo et al. consider an ensemble of agents who has to make a *collective decision* while playing the capital-dependent Parrondo's games, for parameters described in Sect. 2.1. In the form of collective Parrondo's games, this would mean that agents will decide as a whole, on which game they should play collectively. Some of the agents who have a capital multiple of $M = 3$ will prefer to play game A, while the rest of the agents will prefer to play game B. This tension of individual interest makes Parrondo's games a suitable candidate as a model for analysing the effective outcome of collective decision. The model is simulated by varying the value of *N*, the number of agents. At every turn, the agents have to decide whether to play game A or B. Every agent in the population *must* play the same game. There are two ways to play (of course there are an infinite number of strategies, but we choose two that are interesting), (i) the random strategy, where the game is chosen randomly with equal probability. As

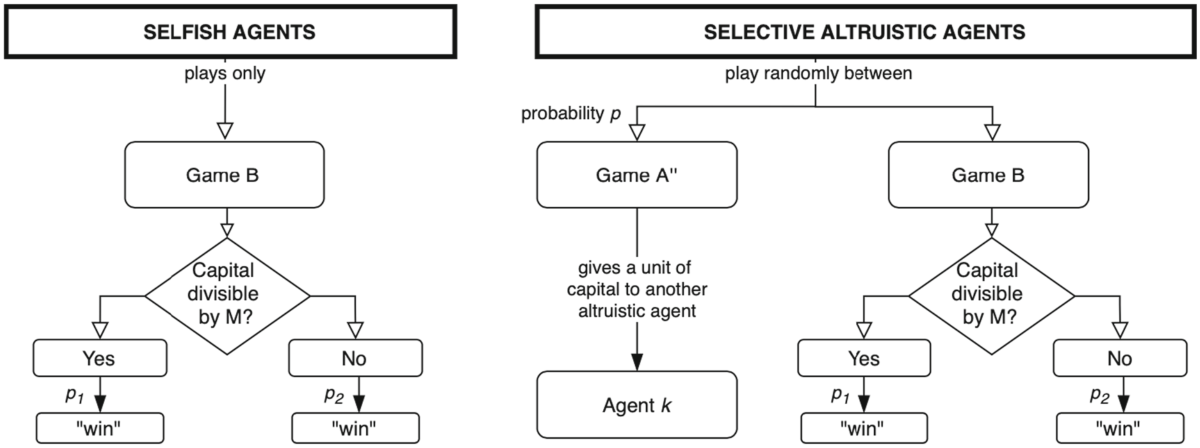


Fig. 14 Selective altruism-selfishness model played by a population N of fully connected agents. Selfish agents only play game B as they do not want to give away their capital. Selective altruistic agents consider two games A$''$ and B, where from time to time they consider giving away a unit of capital to another altruistic agent. Image adapted from Ref. [77]

observed previously, this strategy is a winning strategy for the parameters chosen. (ii) The majority rule strategy, or democratic strategy, where every agent votes for the game giving him/her the highest probability of winning, with the game obtaining the most votes being the collective action. One would think that a democratic majority rule approach would lead to better performance as it optimises the winnings of the majority. However, as it turns out, for large N, strategy (ii) is a losing strategy. To see why, consider the stationary distribution of each game. Let $\pi_i(t)$ be the fraction of players whose capital at time t is $C \equiv i \bmod 3$. With reference to Fig. 3, for $M = 3$, the equation for detailed balance for game A is

$$\begin{pmatrix} \pi_0(t+1) \\ \pi_1(t+1) \\ \pi_2(t+1) \end{pmatrix} = \begin{bmatrix} 0 & 1/2+\varepsilon & 1/2-\varepsilon \\ 1/2-\varepsilon & 0 & 1/2+\varepsilon \\ 1/2+\varepsilon & 1/2-\varepsilon & 0 \end{bmatrix} \times \begin{pmatrix} \pi_0(t) \\ \pi_1(t) \\ \pi_2(t) \end{pmatrix}, \tag{15}$$

which can be written in vector notation

$$\pi(t+1) = \Pi_A \pi(t). \tag{16}$$

Similarly, the same can be written for game B, with

$$\pi(t+1) = \Pi_B \pi(t), \tag{17}$$

where

$$\Pi_B = \begin{bmatrix} 0 & 1/4+\varepsilon & 3/4-\varepsilon \\ 1/10-\varepsilon & 0 & 1/4+\varepsilon \\ 9/10+\varepsilon & 3/4-\varepsilon & 0 \end{bmatrix}. \tag{18}$$

Then, the evolution for strategy (i), the random strategy, is

$$\pi(t+1) = \frac{1}{2}\left[\Pi_A + \Pi_B\right]\pi(t), \tag{19}$$

and for strategy (ii), the majority rule strategy,

$$\pi(t+1) = \begin{cases} \Pi_A \pi(t) & \text{if } \pi_0(t) \geq 1/2 \\ \Pi_B \pi(t) & \text{if } \pi_0(t) < 1/2 \end{cases}. \tag{20}$$

The winning probability in each game is

$$p^A_{\text{win}}(t) = \frac{1}{2} - \varepsilon, \tag{21}$$

$$p^B_{\text{win}}(t) = \frac{1}{10}\pi_0(t) + \frac{3}{4}\left(1 - \pi_0(t)\right) - \varepsilon, \tag{22}$$

and the average capital $\langle C(t)\rangle$ per agent evolves as

$$\langle C(t+1)\rangle = \langle C(t)\rangle + 2p^{\mu}_{\text{win}}(t) - 1, \tag{23}$$

where $p^{\mu}_{\text{win}}(t)$, for $\mu \in \{A, B\}$ depending on the strategy played at turn t. By recursively performing Eq. 23, for the respective game, the long-term effect would be the loss in capital per player. Intuitively, majority rule goes against cooperation (discussed in Sect. 2.3 to have a net positive gain), where agents think of the long-term benefit over short-term gain. In fact, it is no surprise that if $N = 1$, the group that has a single agent will have the highest capital per agent. The single agent will always play the game that gives him/her the highest chance of winning. However, as N increases, the capital per agent decreases.

Another form of collective decision-making is a dictatorship. In a dictatorship, a single agent plays the role of a dictator, and the other agents are "citizens". The dictator chooses the game to play, and the citizens must accept that decision. It is thus logical that the dictator will flourish in this case as he/she can always choose a strategy that best benefits his/her chance of winning. In Ref. [85], a group of $N = 10$ agents play this form of *dictatorship* decision-making game. It can be shown that the net capital gain per turn of the dictator is $g_d = \frac{12}{37} \approx 0.324$, while the citizens have a net capital gain per turn of $g_c = \frac{628{,}224}{13{,}685{,}449} \approx 0.0459$. While the citizens' gain is much lesser than the dictator, counter-intuitively, a dictatorship does lead to a net positive gain for citizens. Additionally, the net capital gain per turn for a citizen under dictatorship is higher than the citizen simply playing a random sequence of games, $g_{rand} \approx 0.0262$. The performance of the whole population under dictatorship is given by

$$g = \frac{g_d + (N-1)g_c}{N}. \tag{24}$$

One would suggest that a way to ensure all agents end up with equal gains is to rotate the agent that gets to play the role of a dictator or to allow the "poorest" citizen to be the dictator at each turn. However, this is not the case because there will always be a single player with the role of the dictator, while the rest play the role of citizens. Thus, the net gain per turn can never exceed g in Eq. 24.

The third option for decision-making is to try improve the performance of the whole population by reducing the decision-making group to a small subset of the population, or an "oligarchy". The oligarchs make a decision by performing a vote, with a threshold, as the democratic strategy discussed earlier. However, simulations performed by Parrondo et al. showed that the optimal size of an oligarch is the whole population. The reason being increasing the size of the oligarch has two effects (1) the oligarch's weight in the entire population increases, improving the net capital gain per turn for a larger group, and (2) the sequence of games approach '...AAB...', which has a positive net capital gain per turn in a capital-dependent Parrondo's games as well.

In the same vein, Ma et al. [87] examine group Parrondo's games as a means to model information transmission to observe the outcome from herd effect in a social network. The dynamics of the model is to employ the use of discrete-time Markov chains in a partial information setting. Consider a group of agents who enters a casino to play slot machines. The agents form a closed ring of N nodes. Each agent has two slot machines C and D to play from. Machine C is installed with either game A or B, and machine D is installed with the game not on machine C. The games A and B are the capital-dependent games discussed in Sect. 2.1. All agents know that the mapping of the slot machines (C,D) can either be (A,B) or (B,A). Furthermore, between the agents, they can exchange information with only nearest immediate neighbours. At each time step, each agent receives the following information:

(a) the machine he/she played in the previous round;
(b) the outcome (win or lose) of the game from the previous round;
(c) the machine his/her nearest neighbours played in the previous round; and
(d) the outcome of the game played by his/her nearest neighbours in the previous round.

The agents *do not* and *are not* keeping track of anyone's capital, including their own. This ensures that the information that an agent receives is minimal; otherwise, the agent can always play the machine that gives the highest payoff. Each agent can adopt one of the two strategies. (i) "Avoid the loser"—agent will play the slot machine not played by the loser(s) when there is no ambiguity; otherwise, the agent will continue playing the same slot machine in the previous game. (ii) "Follow the winner"—agent will play the slot machine played by the winner(s) when there is no ambiguity; otherwise, the agent will continue playing the same slot machine in the previous game. For example, suppose the neighbours of agent i lost in the last game and both used machine C, then i will choose to use machine D in this round. If there is no clear loser to avoid, then i will keep playing the same slot machine from the previous round. In the event that the outcome is always ambiguous such that all agents keep playing their previous slot machines, this is a losing outcome as CCC...C and DDD...D are both losing games.

A Markov chain is set up to perform numerical simulations of the overall effect of this sharing of information. Ma et al. showed that if all agents use the strategy "Follow the winner", this results in a losing strategy. Agents become single-minded, resulting in only one of the two games being played in the long run, without switching (see Fig. 15a). As discussed earlier, this leads to negative returns. Intriguingly, when all agents use the strategy "Avoid the loser", agents do not end up playing the same slot machine. However, the agents also do not switch between the slot machines (see Fig. 15b), another losing outcome. Thus, both strategies (models of herd behaviour) lead to losses. When agents employ a mix of both strategies, the expected capital gain increases over time. This shows that herd behaviour on a simple social network can be detrimental to the progress of the group, whereas randomly switching strategies executed individually without communication may result in progress. Herd behaviour may result in agents making decisions they conclude (or assume) will benefit the group, when in fact, it results in the lack of flow of discourse. This is likened to extreme consensus or polarisation, in both cases, a detriment to the group. If instead, information is allowed to flow and individuals continue to make a decision beneficial to the individual, then this seemingly selfish decision does collectively lead to progress.

4 Conclusions and outlook

The use of Parrondo's games in modelling social dynamics has shed light on many important phenomena that can be beneficial in understanding cooperation and competition, wealth redistribution and welfare, and in information flow and decision-making. Parrondo's paradox has also been applied to model simple social networks. In fact, only the nearest neighbour networks, BA networks, and fully connected networks have been used as models to simulate collective Parrondo's games. However, there are many other forms of network that allow for social interactions. With the rise in social media as a means to transmit information, it has become an integral part of determining social dynamics. For example, the Internet communication network is a scalable multi-agent system which is a gigantic network of communication. The network used by various social media is also different. Social interactions on Facebook is a bidirected network, and two agents who are "friends" can see each other's posts and influence each other's information in-take. Social interactions on Twitter are a directed network. It works on a "follower" framework, where an user sees the posts of the people he/she follows, but the person will not see his/her posts if the person does not follow back. Social interactions on Reddit are more complicated, as it takes on more dimensions. The various topologies of networks may present new applications of Parrondo's paradox in social dynamics and open up new dimensions of research.

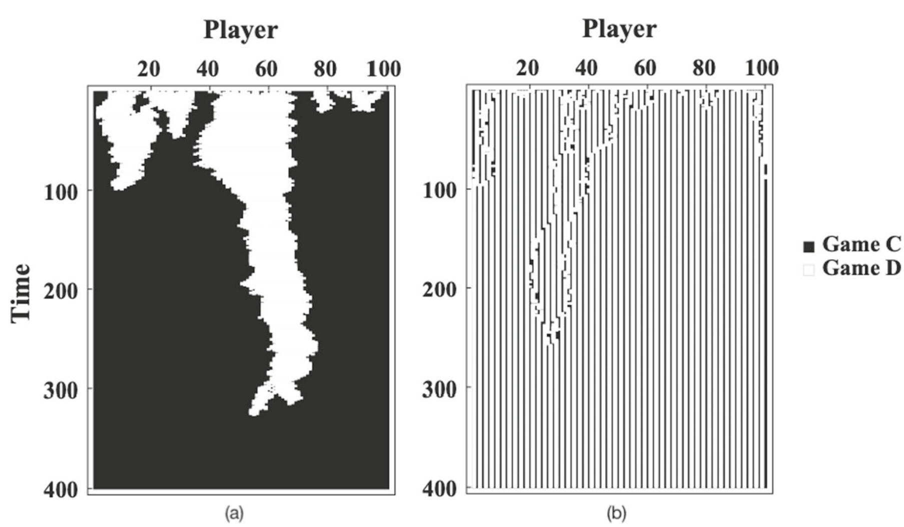


Fig. 15 **a** Outcome of all agents playing the strategy "Follow the winner", resulting in all agents playing the same slot machine C over time. **b** Outcome of all agents playing the strategy "Avoid the loser", resulting in all agents playing different slot machines, but not switching between the slot machines over time. Both strategies are losing strategies. Image reproduced from Figs. 3b and 4b of Ref. [87] with explicit permission.

Another area of development is to broaden the potential impact of real-world data, for example, in performing parameter fitting. This may allow social dynamics to be modelled in a more realistic manner. It is worth noting that most work performed thus far treat social behaviour as discrete units, modelled by discrete "capital" from Parrondo's games. This has led to deterministic outcomes with little sensitivity to initial conditions. However, most social outcomes fall on a continuous spectrum with many deciding features. All these point to the fact that there remains potential for the continuous analogue of Parrondo's paradox to fill this gap in research. The employment of continuum may reveal important nonlinear phenomena and even chaotic behaviour due to perturbations that fall outside of the bounded confidence of initial conditions. This is closely related to the field of opinion dynamics [88–91]—another aspect of social dynamics where computational tools are used to predict the effect of influence in a multi-agent model [92, 93]. Recent interest in how opinions can shape decision-making has renewed interest in how social network structures can reflect the emergence of herd behaviour in political discourse [94, 95], disease spreading [96, 97] and the vaccination dilemma [98–100]. To motivate the potential application of Parrondo's paradox in these highlighted cases, we take the vaccination dilemma as a way of example. A deliberate decision made by individuals not to vaccinate for an infectious disease may be viewed as a "losing" outcome for society as it does not attain the required level of herd immunity. In the event that a pandemic (which may not be entirely related to the vaccination for the aforementioned disease) strikes, this may lead to an outbreak and possibly deaths—another "losing" outcome. We hypothesise that this may then lead to a surge in vaccination uptake, which can be viewed as a "winning" outcome.

Finally, we highlight the theoretical application of Parrondo's paradox in interacting systems. For example, the Ising model is a many-body system of interacting particles, used to predict phase transitions in statistical mechanics [101, 102]. The Ising model can be used to predict inter-particle interaction and interactions with the environment. Thus, it may be useful to apply the Ising model to predict transitions in social behaviour by using Parrondo's games as a control mechanism to the dynamics of these interactions.

In this review article, we have discussed the applications of Parrondo's paradox to social dynamics—in observing important trends such as voting, redistribution of resources, cooperation and competition, communication, improved welfare, as well as survival of the weakest through evolutionary behaviour. As the field of Parrondo's paradox progresses, we are likely to observe increased applicability and accuracy in using it to model complex social interactions for predicting group behaviour. Similarly, greater familiarity with social interactions will lead to informed modelling, which in turn can lead to advancements in research pertaining to Parrondo's paradox.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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