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Parrondo effect: Exploring the nature-inspired framework on periodic functions

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ABSTRACT

Recently, a population model has been analyzed using the framework of Parrondo's paradox to explain how behavior-switching organisms can achieve long-term survival, despite each behavior individually resulting in extinction. By incorporating environmental noise, the model has been shown to be robust to natural variations. Apart from the role of noise, the apparent ubiquity of quasi-periodicity in nature also motivates a more comprehensive understanding of periodically-coupled models of Parrondo's paradox. Such models can enable a wider range of applications of the Parrondo effect to biological and social systems. In this paper, we modify the canonical Parrondo's games to show how the Parrondo effect can still be achieved despite the increased complexity in periodically-noisy environments. Our results suggest the extension of Parrondo's paradox to real-world phenomena strongly subjected to periodic variations, such as ecological systems experiencing seasonal changes, disease in wildlife and humans, or resource management.

1. Introduction

Parrondo's paradox refers to the surprising outcome wherein a winning strategy is obtained by playing two individually losing games in a periodic or random manner. J.M.R Parrondo first introduced a pair of mathematical games of chance, now commonly termed the capital-dependent Parrondo's games, as an abstraction of flashing Brownian ratchets in one-dimensional potentials [1,2]. In these flashing ratchets, toggling a spatially-periodic potential on and off can produce a net drift of diffusive particles, despite there being no large-scale gradient in the potential. Since then, variants of Parrondo's games, such as the history-dependent [3] and cooperative games [4], have inspired numerous works on their mathematical properties and potential applications [5–17].

The counter-intuitive possibility of combining two unfavorable strategies into a favorable one has attracted much attention since its conception. To date, Parrondo's paradox has been greatly useful in enriching our understanding of a wide range of physical phenomena across different fields. In physics, it has been used to understand drifts in granular and diffusive flow [18,19], and in biophysics, it has been useful in the modeling of molecular motors [20]. There are also applications in quantum game theory [21–24] and engineering optimization [25,26]. In systems chemistry, greater product yields are predicted to be achievable by exploiting the Parrondo effect through thermal cycling [27]. The paradox has found numerous applications in life science [28–31], ecology and evolutionary biology [32–34] and social dynamics [35–39].

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In a series of recent papers [40,41], the population dynamics of an environmentally constrained species exhibiting both nomadic and colonial behaviors have been analyzed under the framework of Parrondo's paradox. While both colonialism and nomadism individually lead to population extinction, it was demonstrated that paradoxical proliferation in the long-term is feasible through time-based alternation between the two behaviors. In addition to the Allee effect, the population model also included noise perturbations acting on the environmental carrying capacity, to emulate variations in habitat conditions. This inclusion of simulated noise had demonstrated the robustness of the model and the observed survival–extinction results to environmental noise, which is expected to be ubiquitous in the real world.

The ubiquity of noise, in particular large variations that are periodic or quasi-periodic on the relevant time scales, in natural systems motivate further study. A large range of ecological phenomena manifest themselves with some form of periodicity, such as predator–prey populational oscillations, and patterns induced by seasonal variations in temperature and precipitation, which represent some of the strongest sources of external influence on wildlife [42–45]. In human society, disease dynamics and tourism [46–49] likewise experience periodic variations on a seasonal and yearly timescale, not dissimilar to natural phenomena. In the aforementioned population dynamics system [40,41], the focus was on investigating the effects of environmental noise on dynamical behavior. By necessity, the theoretical treatment and analysis was case-specific, and is not easily generalizable to other areas and models. It can be greatly useful to examine the effects of noise in a more general, abstracted fashion.

In the present study, we investigate the effects of periodic variations on the capital-dependent Parrondo's paradox. In the following sections, we modify the canonical game pair such that it incorporates sinusoidal variations on key parameters; we then examine different such combinations, demonstrating the resulting behavior and the robustness of the Parrondo effect amidst the increased dynamical complexity. Our results suggest Parrondo's games, suitably modified, to be potentially applicable to systems strongly subjected to time-periodic variations.

2. Game model

The paradox can be illustrated using biased coins. To review the theoretical basis of the canonical paradox, we first consider two coin-flipping games, A and B, as introduced in Ref. [2]. In game A, a player flips a biased coin that has a probability (p_1) of winning of less than half. Game B is a capital-dependent game and consists of two biased coins. If the capital is a multiple of an integer $M = 3$, the player flips a coin with a winning probability of p_2. Otherwise, he flips a coin with a winning probability of p_3. The parameter values are usually taken to be

$$p_1 = \frac{1}{2} - \epsilon, \quad p_2 = \frac{1}{10} - \epsilon, \quad p_3 = \frac{3}{4} - \epsilon, \tag{1}$$

for $\epsilon = 0.005 > 0$. In such a configuration, the two games A and B are losing when individually played, but a stochastically mixed game can be winning.

To find the probability distribution across capital states (c **mod** $3 = 0, 1, 2$, where c is the capital) in the long run, we may construct transition matrices to represent these games:

$$\boldsymbol{P}_A = \begin{bmatrix} 0 & p_1 & 1-p_1 \\ 1-p_1 & 0 & p_1 \\ p_1 & 1-p_1 & 0 \end{bmatrix}, \quad \boldsymbol{P}_B = \begin{bmatrix} 0 & p_2 & 1-p_2 \\ 1-p_3 & 0 & p_3 \\ p_3 & 1-p_3 & 0 \end{bmatrix}. \tag{2}$$

The transition matrix for uniformly-mixed stochastic games is then

$$\begin{aligned} \boldsymbol{P}_S &= \frac{1}{2}\left(\boldsymbol{P}_A + \boldsymbol{P}_B\right) \\ &= \frac{1}{2}\begin{bmatrix} 0 & p_1+p_2 & 2-p_1-p_2 \\ 2-p_1-p_3 & 0 & p_1+p_3 \\ p_1+p_3 & 2-p_1-p_3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0.295 & 0.705 \\ 0.38 & 0 & 0.62 \\ 0.62 & 0.38 & 0 \end{bmatrix}, \end{aligned} \tag{3}$$

with the corresponding stationary distribution vector

$$\boldsymbol{\pi} = \begin{bmatrix} 0.345 & 0.254 & 0.401 \end{bmatrix}. \tag{4}$$

The long-term probability of winning, namely $[p_1 + \pi_1 p_2 + (1-\pi_1)p_3]/2 = 0.508 > 1/2$ where π_1 is the first element of $\boldsymbol{\pi}$, indicates a higher probability of winning than losing, hence the paradox.

2.1. Periodically-varying winning probabilities

The original probabilities of winning in the capital-dependent Parrondo's paradox, as given in Eq. (1), are now modified to be

$$p_i = p_i' - f(n), \qquad \text{for } i = 1, 2, 3, \tag{5}$$

where a periodic noise function $f(n)$ is employed to replace the fixed bias parameter ϵ, and p'_i are the base probabilities around which variations from $f(n)$ occur. In our simulations, we use sinusoidal $f(n)$ for simplicity, and we examine different base probabilities p'_1, p'_2 and p'_3 to demonstrate the resulting sinusoidal Parrondo effect across the available parameter space.

Two forms of $f(n)$ are used for demonstrative purposes, to govern the change in winning probabilities:

$$f(n) = C \cdot \sin\left(\alpha\pi \frac{n}{N}\right), \tag{6}$$

$$f(n) = C \cdot \sin\left(\beta\pi \frac{n}{N}\right) \cdot \sin\left(\gamma\pi \frac{n}{N}\right), \tag{7}$$

for amplitude C and frequencies α, β and γ. As implied in Eqs. (5), (6) and (7), it is apparent that the winning probabilities of the capital-dependent Parrondo's games now vary with the game round n.

2.2. Analytical approach

With a given noise function $f(n)$, the transition matrices $\boldsymbol{P}(n)$ for both games A and B become

$$\boldsymbol{P}_A(n) = \begin{bmatrix} 0 & p'_1 - f(n) & 1 - p'_1 + f(n) \\ 1 - p'_1 + f(n) & 0 & p'_1 - f(n) \\ p'_1 - f(n) & 1 - p'_1 + f(n) & 0 \end{bmatrix}, \tag{8}$$

$$\boldsymbol{P}_B(n) = \begin{bmatrix} 0 & p'_2 - f(n) & 1 - p'_2 + f(n) \\ 1 - p'_3 + f(n) & 0 & p'_3 - f(n) \\ p'_3 - f(n) & 1 - p'_3 + f(n) & 0 \end{bmatrix}. \tag{9}$$

Likewise, we have the transition matrix for uniformly-mixed stochastic games as $\boldsymbol{P}_S = (\boldsymbol{P}_A + \boldsymbol{P}_B)/2$. The cumulative mean capital K after n games is then

$$\begin{aligned} K &= \boldsymbol{C}_0\boldsymbol{Q}(1)\boldsymbol{1} + \boldsymbol{C}_0\boldsymbol{P}(1)\boldsymbol{Q}(2)\boldsymbol{1} + \cdots + \boldsymbol{C}_0\boldsymbol{P}(1)\boldsymbol{P}(2)\cdots\boldsymbol{P}(n-1)\boldsymbol{Q}(n)\boldsymbol{1} \\ &= \boldsymbol{C}_0 \left[\sum_{k=1}^{n} \left(\prod_{\ell=1}^{k-1} \boldsymbol{P}(\ell)\right) \boldsymbol{Q}(k)\right] \boldsymbol{1}, \end{aligned} \tag{10}$$

where $C_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ is the initial distribution vector, $\boldsymbol{1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is a column vector of ones, and matrix $\boldsymbol{Q}(n)$ is defined as the Hadamard product $\boldsymbol{Q}(n) = \boldsymbol{P}(n) \circ \boldsymbol{W}$, with payoff matrix

$$\boldsymbol{W} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Above $\boldsymbol{P}$ can be set to be $\boldsymbol{P}_A$, $\boldsymbol{P}_B$, or $\boldsymbol{P}_S$ to model game A, game B, and stochastically mixed games respectively. Let us now consider that $f(n)$ is periodic with period N_0. This periodicity enables a convenient evaluation of the long-term rate of profit. Let us denote the compounded transition matrix over the period of N_0 as $\boldsymbol{H} = \boldsymbol{P}(1)\boldsymbol{P}(2)\cdots\boldsymbol{P}(N_0)$ and let $\boldsymbol{\pi}$ be the stationary distribution of $\boldsymbol{H}$. The long-term rate of capital increase R can then be computed as

$$R = \frac{1}{N_0}\left[\boldsymbol{\pi}\boldsymbol{Q}(1)\boldsymbol{1} + \boldsymbol{\pi}\boldsymbol{P}(1)\boldsymbol{Q}(2)\boldsymbol{1} + \cdots + \boldsymbol{\pi}\boldsymbol{P}(1)\boldsymbol{P}(2)\cdots\boldsymbol{P}(N_0-1)\boldsymbol{Q}(N_0)\boldsymbol{1}\right] \tag{11}$$

$$= \frac{\boldsymbol{\pi}}{N_0}\left[\sum_{k=1}^{N_0} \left(\prod_{\ell=1}^{k-1} \boldsymbol{P}(\ell)\right) \boldsymbol{Q}(k)\right] \boldsymbol{1}, \tag{12}$$

which is similar to the calculation of K, with the important difference that the stationary distribution $\boldsymbol{\pi}$ over the period of N_0 rounds is used instead of the initial distribution C_0.

2.3. Analytical demonstration

As a motivating example, let us consider a time-varying sinusoidal noise function superposed on a fixed bias ϵ, such that $f(n) = \epsilon\,[1 + g(n)]$ with $g(n) = \sin(4\pi n/N)$, $N = 100$ and $\epsilon = 0.005$. The transition matrix for stochastically mixed games is

$$\boldsymbol{P}(n) = \begin{bmatrix} 0 & 0.295 - g(n) & 0.705 + g(n) \\ 0.38 + g(n) & 0 & 0.62 - g(n) \\ 0.62 - g(n) & 0.38 + g(n) & 0 \end{bmatrix}. \tag{13}$$

It can be computed that the stationary distribution vector is $\boldsymbol{\pi} = \begin{bmatrix} 0.345 & 0.254 & 0.401 \end{bmatrix}$, and the long-term rate of capital increase $R > 0$. That is, the game is winning. Note $\boldsymbol{\pi}$ differs only slightly — beyond the decimal places shown here — from that of the noiseless games presented earlier in Eq. (4), as the noise amplitude is small.

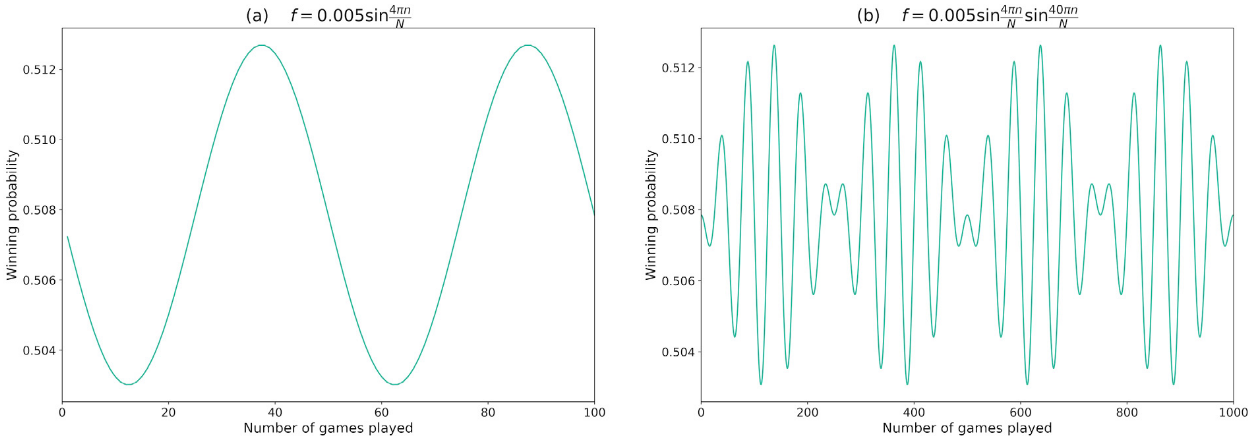

Fig. 1. Winning probabilities of different sinusoidal functions plotted against the number of games played, n. (a) $f = 0.005\sin\left(\frac{4\pi n}{N}\right)$ across 100 games played and (b) $f = 0.005\sin\left(\frac{4\pi n}{N}\right)\sin\left(\frac{40\pi n}{N}\right)$ across 1000 games played. Parameters are $p'_1 = 0.495$, $p'_2 = 0.095$ and $p'_3 = 0.745$, known to be originally losing.

We may also examine the winning probability. Given a transition matrix $\boldsymbol{P}(n)$ subjected to a periodic function $f(n)$, the winning probability, $P_{\text{win}}(n)$, after n rounds, is

$$P_{\text{win}}(n) = \boldsymbol{\pi}\boldsymbol{P}(1)\boldsymbol{P}(2)\cdots\boldsymbol{P}(n-1)\begin{bmatrix}\boldsymbol{P}(n)_{12}\\ \boldsymbol{P}(n)_{23}\\ \boldsymbol{P}(n)_{31}\end{bmatrix}, \tag{14}$$

where $\boldsymbol{P}(n)_{ij}$ refers to the entry of $\boldsymbol{P}(n)$ in the ith row and the jth column.

We apply the above analysis to two different sinusoidal noise functions, $f = 0.005\sin(4\pi n/N)$ and $f = 0.005\sin(4\pi n/N)\sin(40\pi n/N)$, and illustrate their corresponding winning probabilities for the modified sinusoidal Parrondo's games in Fig. 1. Clearly, the probabilities of winning for both functions are always greater than 1/2, despite the periodic variations having no net bias over their periods. In combination with the previous example, this suggests that a Parrondo-paradoxical outcome is indeed realizable in the noise-coupled games, which we confirm in the following subsections with more extensive simulations.

3. Results

Numerical simulations implementing the periodic noise-coupled Parrondo's games were performed using *Python*. The standard Python `random` package, with the core generator being a Mersenne Twister pseudo-random number generator providing a sufficient float precision and period [50], was used to generate random floats uniformly in the range [0.0, 1.0), which were used to stochastically determine the game to be played at each round, and to determine the outcome of the games. The expectation values of each simulation performed were averaged over 10^6 trials to ensure convergence of results. For $f(n) = C\cdot\sin\left(\beta\pi\frac{n}{N}\right)\cdot\sin\left(\gamma\pi\frac{n}{N}\right)$, more game rounds were played ($n = 1000$) to demonstrate the long-term behavior. The simulation results are in agreement with the analytical calculations presented in Section 2.2.

3.1. Parrondo effect with $f(n) = C\sin\left(\alpha\pi\frac{n}{N}\right)$

Simulation results using the sinusoidal noise function $f(n) = C\sin\left(\alpha\pi\frac{n}{N}\right)$ are presented in Figs. 2 and 3. Clearly, with the introduction of the noise function, both games A and B remain individually as losing games, as observed in Fig. 2.

The Parrondo's paradox then manifests when the two losing games are combined periodically — in the order of two games of A followed by two games of B — and randomly, with no order of sequence. In both cases, winning outcomes, as characterized by a long-term gain in capital, are observed, as shown in Fig. 3. The original results obtained in [2] are also plotted for a direct comparison.

3.2. Robustness with $f(n) = C\cdot\sin\left(\beta\pi\frac{n}{N}\right)\cdot\sin\left(\gamma\pi\frac{n}{N}\right)$

To demonstrate the robustness of the observed Parrondo effect in different noise environments, we now attempt to use the more complicated sinusoidal noise function $f(n) = C\cdot\sin\left(\beta\pi\frac{n}{N}\right)\cdot\sin\left(\gamma\pi\frac{n}{N}\right)$. Simulation results in Fig. 4 confirm that games A and B remain individually as losing games while a random or periodic mix of the games yield the Parrondo effect, as observed in Fig. 5. Across different values of β and γ, it is observed in Figs. 6 and 7 that the characteristics of

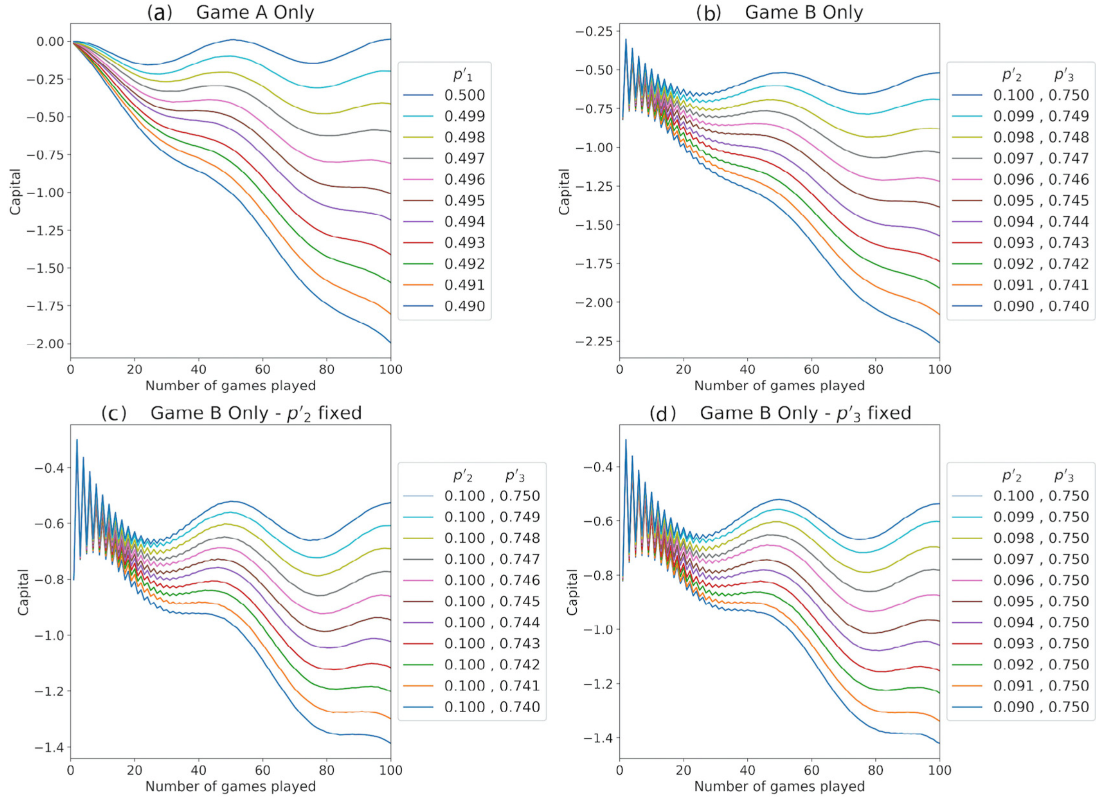

Fig. 2. Numerical simulations of games A and B individually, using the noise function $f(n) = 0.005\sin\left(4\pi\,\frac{n}{N}\right)$. (a) Game A only with different p_1' values. (b) – (d) Game B only with different combinations of p_2' and p_3'. The results were averaged over 10^6 trials.

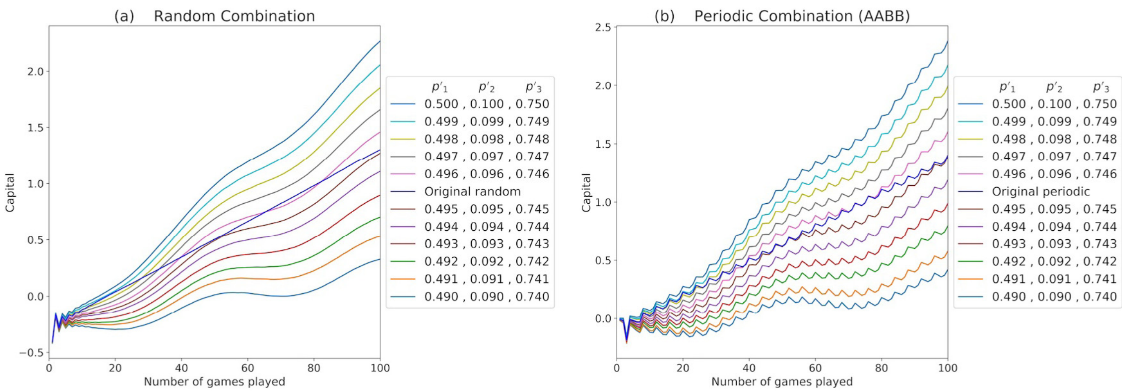

Fig. 3. Parrondo's paradox is observed in both (a) random combination of games A and B and (b) periodic combination of games A and B (order AABB). Results of the original games for both random and periodic combinations are also plotted for clearer comparison. The noise function $f(n) = 0.005\sin\left(4\pi\,\frac{n}{N}\right)$ is used.

Parrondo's paradox are still retained: periodic and random combinations of individually losing games A and B still lead to winning outcomes. In addition, the qualitative zig-zag shape characteristic of the periodic combination in Fig. 3(b) is also observed in Figs. 5(b) and 7(b), when zoomed in. These results, taken together, confirm the persistence of the Parrondo effect, and suggest that the framework of Parrondo's games may be applicable in natural systems subjected to ubiquitous periodic noise. We note it may be interesting to consider the uniformity in behavior of the Parrondo effect with periodic variations, with respect to the closed-interval range of $f(n)$, as n grows large; this motivates future work.

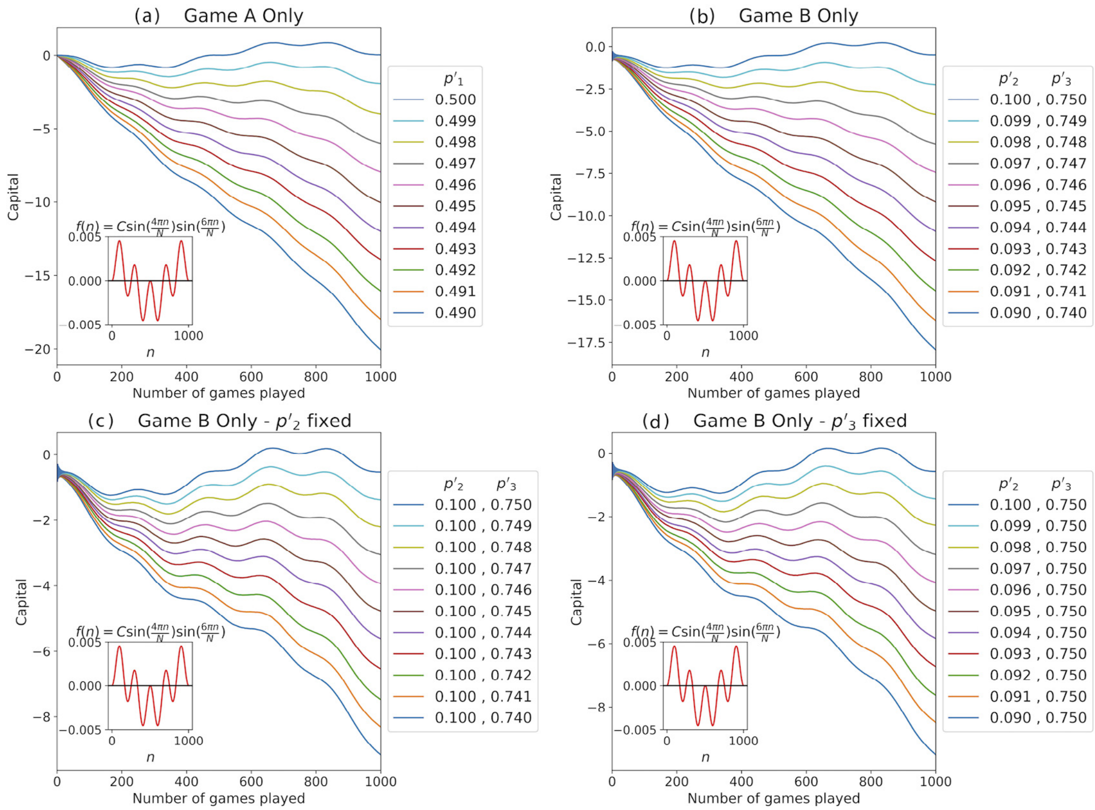

Fig. 4. Numerical simulations of games A and B individually using the noise function $f(n) = 0.005\sin\left(4\pi\frac{n}{N}\right)\cdot\sin\left(6\pi\frac{n}{N}\right)$.

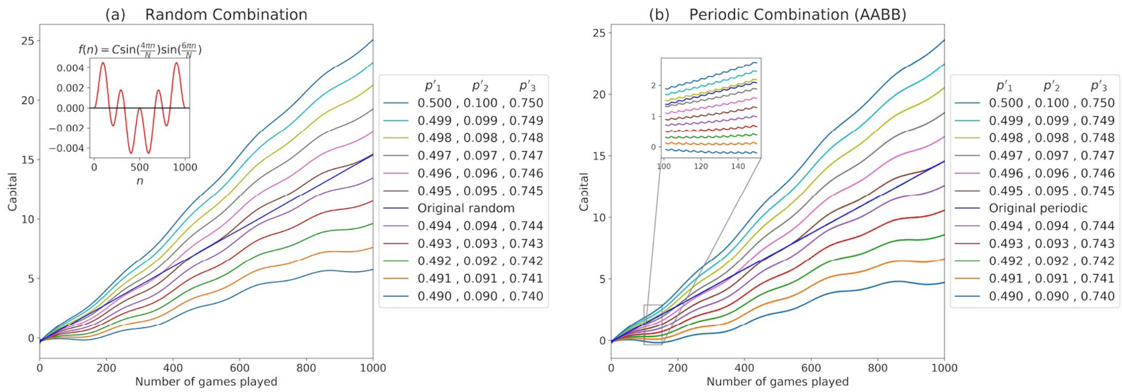

Fig. 5. Parrondo's paradox is observed in both (a) random combination of games A and B and (b) periodic combination of games A and B (order AABB). The noise function $f(n) = 0.005\sin\left(4\pi\frac{n}{N}\right)\cdot\sin\left(6\pi\frac{n}{N}\right)$ is used.

4. Conclusion

Motivated by the ubiquity of periodicity and quasi-periodicity in nature, such as seasonal variations in temperature and predator–prey populational oscillations, we have investigated a model of Parrondo's paradox subject to periodic variations in game probability parameters. By demonstrating the existence of the Parrondo effect in these games, and examining their response and robustness to different imposed noise variations, the present study contributes in

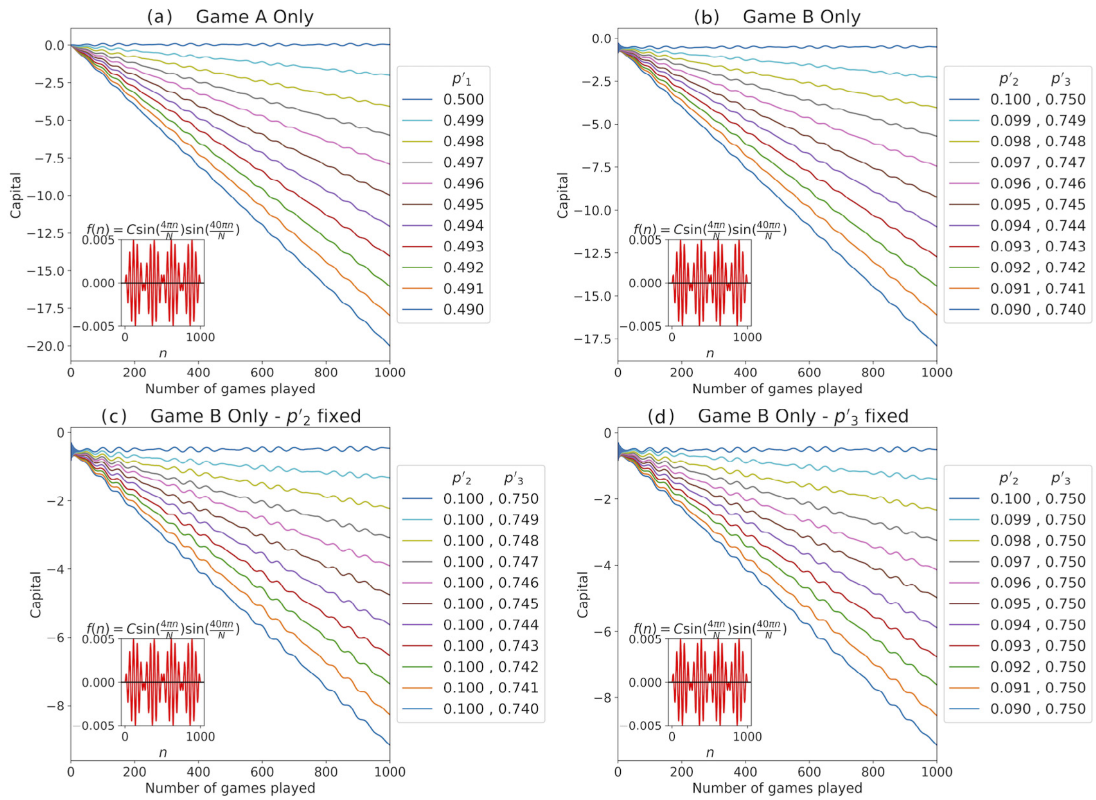

Fig. 6. Numerical simulations of games A and B individually using the noise function $f(n) = 0.005\sin\left(4\pi\,\frac{n}{N}\right) \cdot \sin\left(40\pi\,\frac{n}{N}\right)$.

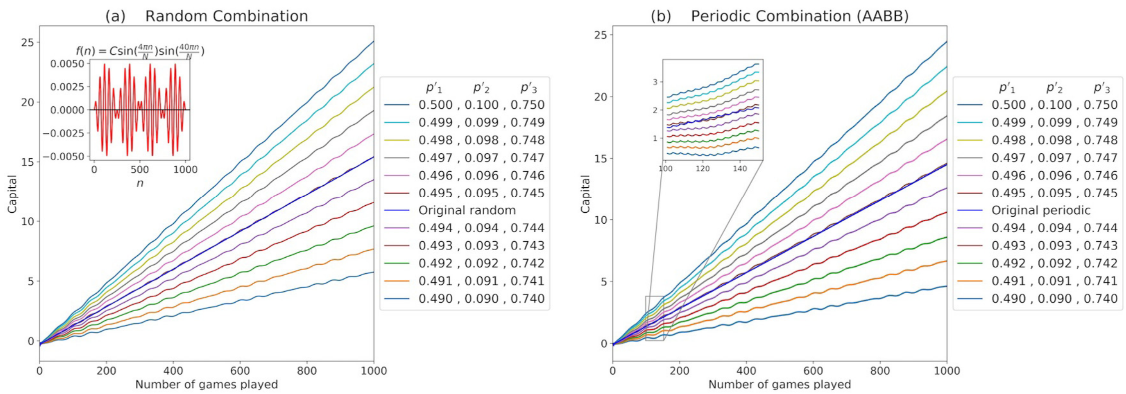

Fig. 7. Parrondo's paradox is observed in both (a) random combination of games A and B and (b) periodic combination of games A and B (order AABB). The noise function $f(n) = 0.005\sin\left(4\pi\,\frac{n}{N}\right) \cdot \sin\left(40\pi\,\frac{n}{N}\right)$ is used.

extending Parrondo's games to modeling realistic ecological and biological scenarios, in which variations of approximate periodic nature frequently manifest. For instance, in disease dynamics, seasonal variations in disease transmission is often synchronized with human behavioral patterns, most prominently travel and festivities, that exhibit a seasonal rhythm; and, not dissimilarly, behavioral patterns in wildlife are often synchronized with climate and food availability patterns, including extreme feats of migration, and mating cycles.

CRediT authorship contribution statement

Shuyi Jia: Software, Methodology, Validation, Formal analysis, Investigation, Data curation, Writing - original draft, Writing - review & editing, Visualization. **Joel Weijia Lai:** Software, Methodology, Validation, Formal analysis, Investigation, Data curation, Writing - original draft, Writing - review & editing, Visualization. **Jin Ming Koh:** Methodology, Validation, Formal analysis, Investigation, Writing - original draft, Writing - review & editing, Visualization. **Neng Gang Xie:** Validation, Formal analysis, Investigation, Writing - review & editing, Visualization. **Kang Hao Cheong:** Conceptualization, Methodology, Validation, Formal analysis, Investigation, Resources, Data curation, Writing - original draft, Writing - review & editing, Visualization, Supervision, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] Gregory P. Harmer, Derek Abbott, A review of Parrondo's paradox, Fluct. Noise Lett. 2 (02) (2002) R71–R107.
[2] Gregory P. Harmer, Derek Abbott, Game theory: Losing strategies can win by Parrondo's paradox, Nature 402 (6764) (1999) 864.
[3] Juan M.R. Parrondo, Gregory P. Harmer, Derek Abbott, New paradoxical games based on Brownian ratchets, Phys. Rev. Lett. 85 (24) (2000) 5226.
[4] Raúl Toral, Cooperative Parrondo's games, Fluct. Noise Lett. 1 (01) (2001) L7–L12.
[5] Wayne Wah Ming Soo, Kang Hao Cheong, Parrondo's paradox and complementary Parrondo processes, Physica A 392 (1) (2013) 17–26.
[6] Wayne Wah Ming Soo, Kang Hao Cheong, Occurrence of complementary processes in Parrondo's paradox, Physica A 412 (2014) 180–185.
[7] Kang Hao Cheong, Wayne Wah Ming Soo, Construction of novel stochastic matrices for analysis of Parrondo's paradox, Physica A 392 (20) (2013) 4727–4738.
[8] Jose S. Cánovas, María Muñoz Guillermo, Computing the topological entropy of continuous maps with at most three different kneading sequences with applications to Parrondo's paradox, Chaos Solitons Fractals 83 (2016) 1–17.
[9] S. Ethier, Jiyeon Lee, Limit theorems for Parrondo's paradox, Electron. J. Probab. 14 (2009) 1827–1862.
[10] Stewart Ethier, Jiyeon Lee, Parrondo's paradox via redistribution of wealth, Electron. J. Probab. 17 (2012).
[11] Degang Wu, Kwok Yip Szeto, Extended Parrondo's game and Brownian ratchets: Strong and weak Parrondo effect, Phys. Rev. E 89 (2) (2014) 022142.
[12] Ka Wai Cheung, Ho Fai Ma, Degang Wu, Ga Ching Lui, Kwok Yip Szeto, Winning in sequential Parrondo games by players with short-term memory, J. Stat. Mech. Theory Exp. 2016 (5) (2016) 054042.
[13] Kang Hao Cheong, David B. Saakian, Rubina Zadourian, Allison mixture and the two-envelope problem, Phys. Rev. E 96 (6) (2017) 062303.
[14] K.H. Cheong, J.M. Koh, D.J. Yeo, Z.X. Tan, B.O.E. Boo, G.Y. Lee, Paradoxical simulations to enhance education in mathematics, IEEE Access 7 (2019) 17941–17950, http://dx.doi.org/10.1109/ACCESS.2019.2892742.
[15] Bruno Rémillard, Jean Vaillancourt, Combining losing games into a winning game, Fluct. Noise Lett. 18 (01) (2019) 1950003.
[16] S.N. Ethier, Jiyeon Lee, How strong can the Parrondo effect be?, J. Appl. Probab. 56 (4) (2019) 1198–1216.
[17] S.N. Ethier, Jiyeon Lee, How strong can the parrondo effect be? II, 2020, arXiv preprint arXiv:2001.00291.
[18] Anthony Rosato, Katherine J. Strandburg, Friedrich Prinz, Robert H. Swendsen, Why the Brazil nuts are on top: Size segregation of particulate matter by shaking, Phys. Rev. Lett. 58 (10) (1987) 1038.
[19] Ross Pinsky, Michael Scheutzow, Some remarks and examples concerning the transience and recurrence of random diffusions, in: Annales de L'IHP Probabilités et Statistiques, vol. 28, 1992, pp. 519–536.
[20] David Heath, David Kinderlehrer, Michal Kowalczyk, Discrete and continuous ratchets: from coin toss to molecular motor, Discrete Contin. Dyn. Syst. Ser. B 2 (2) (2002) 153–168.
[21] Jishnu Rajendran, Colin Benjamin, Playing a true Parrondo's game with a three-state coin on a quantum walk, Europhys. Lett. 122 (4) (2018) 40004.
[22] Jishnu Rajendran, Colin Benjamin, Implementing Parrondo's paradox with two-coin quantum walks, Royal Soc. Open Sci. 5 (2) (2018) 171599.
[23] Joel Weijia Lai, Kang Hao Cheong, Parrondo's paradox from classical to quantum: A review, Nonlinear Dynam. 100 (2020) 849—861.
[24] Joel Weijia Lai, Kang Hao Cheong, Parrondo effect in quantum coin-toss simulations, Phys. Rev. E 101 (2020) 052212.
[25] Jin Ming Koh, Kang Hao Cheong, Automated electron-optical system optimization through switching Levenberg–Marquardt algorithms, J. Electron Spectrosc. Relat. Phenom. 227 (2018) 31–39.
[26] Kang Hao Cheong, Jin Ming Koh, A hybrid genetic-levenberg marquardt algorithm for automated spectrometer design optimization, Ultramicroscopy 202 (2019) 100–106.
[27] Daniel C. Osipovitch, Carl Barratt, Pauline M. Schwartz, Systems chemistry and Parrondo's paradox: Computational models of thermal cycling, New J. Chem. 33 (10) (2009) 2022–2027.
[28] Kang Hao Cheong, Jin Ming Koh, Michael C. Jones, Paradoxical survival: Examining the parrondo effect across biology, BioEssays 41 (6) (2019) 1900027.
[29] Kang Hao Cheong, Jin Ming Koh, Michael C. Jones, Multicellular survival as a consequence of Parrondo's paradox, Proc. Natl. Acad. Sci. 115 (23) (2018) E5258–E5259.
[30] Kang Hao Cheong, Jin Ming Koh, Michael C. Jones, Do arctic hares play Parrondo's games?, Fluct. Noise Lett. 18 (03) (2019) 1971001, http://dx.doi.org/10.1142/S0219477519710019.

[31] Jin Ming Koh, Neng-gang Xie, Kang Hao Cheong, Nomadic-colonial switching with stochastic noise: subsidence-recovery cycles and long-term growth, Nonlinear Dynam. 94 (2) (2018) 1467–1477.
[32] Zhi-Xuan Tan, Jin Ming Koh, Eugene V. Koonin, Kang Hao Cheong, Predator dormancy is a stable adaptive strategy due to Parrondo's paradox, Adv. Sci. 7 (3) (2020) 1901559.
[33] Zhi-Xuan Tan, Kang Hao Cheong, Periodic habitat destruction and migration can paradoxically enable sustainable territorial expansion, Nonlinear Dynam. 98 (1) (2019) 1–13.
[34] Kang Hao Cheong, Zong Xuan Tan, Neng-gang Xie, Michael C Jones, A paradoxical evolutionary mechanism in stochastically switching environments, Sci. Rep. 6 (2016) 34889.
[35] Ye Ye, Kang Hao Cheong, Yu-wan Cen, Neng-gang Xie, Effects of behavioral patterns and network topology structures on Parrondo's paradox, Sci. Rep. 6 (2016) 37028.
[36] Ye Ye, Xiao Rong Hang, Jin Ming Koh, Jarosaw Adam Miszczak, Kang Hao Cheong, Neng-gang Xie, Ratcheting based on neighboring niches determines lifestyle, Nonlinear Dynam. 98 (3) (2019) 1821–1830.
[37] Ye Ye, Xiao Rong Hang, Jin Ming Koh, Jarosaw Adam Miszczak, Kang Hao Cheong, Neng gang Xie, Passive network evolution promotes group welfare in complex networks, Chaos Solitons Fractals 130 (2020) 109464, http://dx.doi.org/10.1016/j.chaos.2019.109464.
[38] Jin Ming Koh, Kang Hao Cheong, New doubly-anomalous Parrondo's games suggest emergent sustainability and inequality, Nonlinear Dynam. 96 (1) (2019) 257–266.
[39] Jin Ming Koh, Kang Hao Cheong, Emergent preeminence of selfishness: an anomalous Parrondo perspective, Nonlinear Dynam. 98 (2) (2019) 943–951.
[40] Zong Xuan Tan, Kang Hao Cheong, Nomadic-colonial life strategies enable paradoxical survival and growth despite habitat destruction, eLife 6 (2017) e21673.
[41] Kang Hao Cheong, Zong Xuan Tan, Yan Hao Ling, A time-based switching scheme for nomadic-colonial alternation under noisy conditions, Commun. Nonlinear Sci. Numer. Simul. 60 (2018) 107–114.
[42] Olivier Gilg, Ilkka Hanski, Benoit Sittler, Cyclic dynamics in a simple vertebrate predator-prey community, Science 302 (5646) (2003) 866–868.
[43] Charles J. Krebs, Alice J. Kenney, Scott Gilbert, Kjell Danell, Anders Angerbjörn, Sam Erlinge, Robert G. Bromley, Chris Shank, Suzanne Carriere, Synchrony in lemming and vole populations in the Canadian arctic, Can. J. Zool. 80 (8) (2002) 1323–1333.
[44] Kenneth V. Yeargan, Prey and periodicity of Pardosa ramulosa (McCook) in alfalfa, Environ. Entomol. 4 (1) (1975) 137–141.
[45] Sonia Altizer, Andrew Dobson, Parviez Hosseini, Peter Hudson, Mercedes Pascual, Pejman Rohani, Seasonality and the dynamics of infectious diseases, Ecol. Lett. 9 (4) (2006) 467–484.
[46] Scott F. Dowell, Seasonal variation in host susceptibility and cycles of certain infectious diseases, Emerg. Infect. Dis. 7 (3) (2001) 369.
[47] David N. Fisman, Seasonality of infectious diseases, Annu. Rev. Publ. Health 28 (2007) 127–143.
[48] Tom Baum, Seasonality in tourism: Understanding the challenges: Introduction, Tour. Econ. 5 (1) (1999) 5–8.
[49] Lee Jolliffe, Regena Farnsworth, Seasonality in tourism employment: Human resource challenges, Int. J. Contemp. Hosp. Manag. (2003).
[50] Makoto Matsumoto, Takuji Nishimura, Mersenne twister: a 623-dimensionally equidistributed uniform pseudo-random number generator, ACM Trans. Model. Comput. Simul. (TOMACS) 8 (1) (1998) 3–30.